ABSTRACT

Title of Dissertation: The Gauge-Uzawa and Related Projection Finite Element Methods for the Evolution Navier-Stokes Equations

Jae-Hong Pyo, Doctor of Philosophy, 2002

Dissertation directed by: Professor Ricardo H. Nochetto
Department of Mathematics

The Navier-Stokes of incompressible fluids are still a computational challenge. Numerical difficulties arise from the incompressibility constraint, which requires a compatibility condition (discrete inf-sup) between the finite element spaces for velocity and pressure, the nonlinear convection term, and the presence of disparate scales in both space and time. Several projection methods have been introduced for time discretization to circumvent the incompressibility constraint, but suffer from boundary layers. They are either numerical or due to non-physical boundary conditions on pressure.

Recently W. E and J.-G. Liu [7] introduced the gauge method which is a projection type method. The main advantage of the gauge method is its PDE formulation with "artificial" boundary conditions imposed on the non-physical

gauge variable ϕ rather than the pressure. Its chief disadvantage is the implementation of such boundary conditions at the discrete level, because they involve both normal and tangential derivatives of ϕ. Even though E and Liu suggested that either Neumann or Dirichlet boundary conditions can be chosen for ϕ, we disclose below a compatibility constraint for the Dirichlet condition which severely limits its use. We construct 4 time-discrete gauge methods and prove error estimates for both velocity and pressure under realistic regularity assumptions via a variational approach. This improves upon the error analysis of J.G. Liu and C. Wang [30], which relies on asymptotic expansions, requires strong regularity, and yields no estimate for pressure. We introduce variational methods to do accurate boundary computations of normal and tangential derivatives of ϕ in the discrete finite element spaces. The need of such derivatives, however, restricts the applicability of Gauge methods to 2d. Also the computing cost is relatively high, because ϕ requires a higher polynomial degree than pressure and velocity for a stable computation.

In order to overcome these difficulties, we construct the Gauge-Uzawa (GU) method which exhibits several advantages. The first one is that GU is a fully discrete algorithm, and is amenable to a complete and rigorous analysis. In contrast many projection-type methods are formulated as suitable time discretizations but are difficult to study once space is also discretized. Often, the discrete spaces for velocity, pressure, and other artificial variables such as ϕ cannot be chosen arbitrarily. In contrast, GU does not incorporate inconsistencies or incompatibilities between space and time discretizations. The second advantage of GU is in dealing with boundary values. As opposed to the Gauge method, GU does not require any boundary calculation and thus applies to any space dimension.

The third advantage of GU is the relatively low computational cost. Since ϕ is included in the pressure space, which is of lower polynomial degree than the velocity space, GU is more efficient than the Gauge method. This is documented herein with extensive numerical simulations. The last advantage is that GU is unconditionally stable, which extends its applicability to large Reynolds numbers. We prove error estimates for both velocity and pressure under realistic regularity assumptions via a variational approach, extending the ideas and methodology of the semi-discrete scheme to the fully discrete case. All our numerical experiments show that errors of GU are smaller than those of other projection-type scheme.

We study the motion of incompressible fluids driven by thermal effects. The underlying model consists of coupling the Navier-Stokes equations with Boussinesq approximation with the heat equation. We couple GU with a finite element discretization of the heat equation, and implement the system, and test its behavior for several interesting situations (including a benchmark computation by Gresho et al [13]). We finally derive a complete error analysis for the method very much in the spirit of GU for the Navier-Stokes equations.

The Uzawa method is a very well known iteration for the stationary Stokes equations. It consists of a velocity update U_{j+1} involving a solve for the Laplacian, followed by a pressure update, a so-called Richardson update:

$$P_{j+1} = P_j - \alpha \operatorname{div} U_{j+1}.$$

The relaxation parameter $\alpha > 0$ must be taken sufficiently small for the iteration to converge. We prove, as was generally believed, that $\alpha = 1$ is a suitable and simple choice for α. This relies on the elementary, yet new, observation that $\|\operatorname{div} U\|_0 \leq \|\nabla U\|_0$ for all $U \in \mathbf{H}_0^1(\Omega)$. We also extend this result to stable finite element discretizations by showing that the discrete inf-sup constant β is always

< 1 and the rate of convergence is $1 - \beta^2$.

The Gauge-Uzawa and Related Projection Finite Element Methods for the Evolution Navier-Stokes Equations

by

Jae-Hong Pyo

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland at College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2002

Advisory Committee:

Professor Ricardo H. Nochetto, Chairman/Advisor
Professor John E. Osborn
Professor Jian-Guo Liu
Professor Bo Li
Professor Howard Elman

DEDICATION

To parent and my family

ACKNOWLEDGEMENTS

I would like to thank my advisor, professor Ricardo H. Nochetto for his advice, support, and encouragement throughout the duration of the work. His help was not limited to mathematics. He arranged for me to take an English course and provided financial support during the financial crisis in Korea. I especially appreciate him for his guidance in getting me a T.A. position, which was my first job in America.

I wish to thank my parents for their financial support, for praying to God, and for believing in me.

I also would like to thank my family members for their endurance and support for a long time.

Most of all, I thank God for guiding me.

TABLE OF CONTENTS

List of Tables vii

List of Figures viii

1 Introduction 1

1.1 Navier-Stokes Equations 1

1.2 Preliminaries and Assumptions 6

1.3 Benchmark Problems 19

1.3.1 Example: Smooth Solution 19

1.3.2 Example: Singular Solution 21

2 Projection Methods 24

2.1 Chorin Method 24

2.2 Chorin-Uzawa Method 26

2.3 Numerical Results for Chorin and Chorin-Uzawa Methods 29

2.3.1 Experiments with $P_1 - P_1$ Elements 29

2.3.2 Experiments with $P_2 - P_1$ Elements 34

2.3.3 Experiments for Chorin-Uzawa with $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 38

2.3.4 Example : Singular Solution 38

3 Gauge Method **42**

3.1 Motivation of Gauge Method 42

3.2 Time Discretization and Algorithms 45

3.3 Stability 49

3.3.1 Explicit Convection Scheme 50

3.3.2 Semi-Implicit Convection Scheme 52

3.4 A Priori Error Analysis for Velocity of Algorithms 3.1-3.2 with Neumann Condition 53

3.4.1 Semi-Implicit Convection Scheme 55

3.4.2 Explicit Convection Scheme 64

3.5 A Priori Error Analysis for Velocity : Algorithms 3.3-3.4 with Dirichlet Condition 67

3.6 A Priori Error Analysis for Pressure 70

3.7 Conclusion and Numerical Results for Gauge methods 88

3.7.1 Algorithms 3.1 and 3.2 : Neumann Boundary Condition . 89

3.7.2 Algorithm 3.1 : $P_1 - P_1 - P_1$ on Regular Domain 106

3.7.3 Algorithms 3.3 and 3.4 : Dirichlet Boundary Condition . . 108

3.7.4 Example : Singular Solution 124

4 Iterative Solvers for the Stationary Stokes Equations **126**

4.1 Variational computation of boundary differentiations 127

4.2 Space Discretization via Gauge Method 131

4.3 Simulations and Conclusions for the Gauge method 136

4.3.1 Numerical Experiments for Algorithms 4.1-4.2 137

4.3.2 Numerical Experiments for Algorithms 4.3-4.4 148

4.4 Gauge-Uzawa Method for Stationary Stokes 158

4.5 Numerical Experiment for Gauge-Uzawa 163
4.6 Uzawa Method 164

5 Gauge-Uzawa Method for the Navier-Stokes Equations 169
5.1 Motivation of Gauge-Uzawa Method 169
5.2 Stability 173
5.3 Error Estimate for Velocity 175
5.4 Error Estimate for Pressure 192
5.5 Numerical Experiments 222
5.5.1 Example : Smooth Solution on Distorted Mesh (a) in Figure 1.2 222
5.5.2 Example : Smooth Solution on Regular Mesh (b) in Figure 1.2 231
5.5.3 Example : Singular Solution 234
5.5.4 Example : Forward Facing Step 236
5.5.5 Example : Driven Cavity Flow 237

6 Gauge-Uzawa Method for the Evolution Boussinesq Equations 238
6.1 Gauge-Uzawa Method for Boussinesq Equations 240
6.2 Regularity of Boussinesq Equations 241
6.3 Stability 248
6.4 Error Estimate for Velocity and Temperature 251
6.5 Error Estimate for Pressure 269
6.6 Numerical Experiments 276

Bibliography 292

LIST OF TABLES

1.1 The Relation between Refinement Level and Finite Element Structure 22

3.1 Summary of Gauge Methods 88

5.1 The Notations of Error 179

LIST OF FIGURES

1.1 (a) Macro mesh, (b) Refinement of a triangle 21

1.2 Quasi-Uniform and Regular Meshes with 4 Levels of Refinement . 21

1.3 The Computational Mesh for Singular Solution 23

2.1 Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_1 - P_1$ Elements. 30

2.2 Error Functions for Chorin Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963). 31

2.3 Error Functions for Chorin-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963). 31

2.4 Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_1 - P_1$ Elements. 32

2.5 Error Functions for Chorin Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963). 33

2.6 Error Functions for Chorin-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963). 33

2.7 Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_2 - P_1$ Elements. 34

2.8 Error Functions for Chorin Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371). 35

2.9 Error Functions for Chorin-Uzawa Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371). 35
2.10 Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements. 36
2.11 Error Functions for Chorin Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371). 37
2.12 Error Functions for Chorin-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371). 37
2.13 Error Decay of Chorin-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2. 39
2.14 Error Functions of Chorin-Uzawa Method with $\triangle t = h^2$ Chorin-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963). 40
2.15 Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements. 40
2.16 Numerical Solution of Chorin-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903). 41
2.17 Numerical Solution of Chorin Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903). 41

3.1 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements. 90
3.2 Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 91
3.3 Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 91

3.4 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements. 92
3.5 Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 93
3.6 Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 93
3.7 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements. 94
3.8 Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 95
3.9 Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 95
3.10 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements. 96
3.11 Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 97
3.12 Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 97
3.13 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements. 98
3.14 Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 99
3.15 Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 99

3.16 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements. 100
3.17 Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ (DOF = 74,371) Elements. 101
3.18 Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 101
3.19 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements. 102
3.20 Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 103
3.21 Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 103
3.22 Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 104
3.23 Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 105
3.24 Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 105
3.25 Error Decay of Algorithm 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2. 106
3.26 Error Functions of Algorithm 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963). 107
3.27 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements. 108

3.28 Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ (DOF = 24,963) Elements. 109
3.29 Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 109
3.30 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements. 110
3.31 Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 111
3.32 Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963). 111
3.33 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements. 112
3.34 Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 113
3.35 Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 113
3.36 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements. 114
3.37 Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 115
3.38 Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667). 115
3.39 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements. 116

3.40 Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 117

3.41 Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 117

3.42 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements. 118

3.43 Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 119

3.44 Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371). 119

3.45 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements. 120

3.46 Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 121

3.47 Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 121

3.48 Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 122

3.49 Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 123

3.50 Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075). 123

3.51 Error Decay of Gauge Method Algorithm 3.1 with $\triangle t = h$ and $P_2 - P_1 - P_3$ Elements. 124

3.52 Numerical Solution of Gauge Method Algorithm 3.1 with $\triangle t = h$ and $P_2 - P_1 - P_3$ Elements (DOF = 158,119). 125

4.1 $\dfrac{\partial\phi}{\partial\nu} = 0$ at Each Corner Provided $\phi = 0$. 128

4.2 $\dfrac{\partial\phi}{\partial\tau} = 0$ at Each Corner Provided $\dfrac{\partial\phi}{\partial\nu} = 0$. 129

4.3 Difficulty of Variational Formula in 3D 130

4.4 Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_1 - P_1 - P_1$. 138

4.5 Error functions for Algorithm 4.1 with $P_1 - P_1 - P_1$. 139

4.6 Error functions for Algorithm 4.2 with $P_1 - P_1 - P_1$. 139

4.7 Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_1 - P_1 - P_2$ 140

4.8 Error functions for Algorithm 4.1 with $P_1 - P_1 - P_2$. 141

4.9 Error functions for Algorithm 4.2 with $P_1 - P_1 - P_2$. 141

4.10 Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_1$ 142

4.11 Error functions for Algorithm 4.1 with $P_2 - P_1 - P_1$. 143

4.12 Error functions for Algorithm 4.2 with $P_2 - P_1 - P_1$. 143

4.13 Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_2$. 144

4.14 Error functions for Algorithm 4.1 with $P_2 - P_1 - P_2$. 145

4.15 Error functions for Algorithm 4.2 with $P_2 - P_1 - P_2$. 145

4.16 Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_3$. 146

4.17 Error functions of Algorithm 4.1 with $P_2 - P_1 - P_3$ 147

4.18 Error functions for Algorithm 4.2 with $P_2 - P_1 - P_3$. 147

4.19 Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_1 - P_1 - P_1$. 148
4.20 Error functions for Algorithm 4.3 with $P_1 - P_1 - P_1$. 149
4.21 Error functions for Algorithm 4.4 with $P_1 - P_1 - P_1$. 149
4.22 Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_1 - P_1 - P_2$ 150
4.23 Error functions for Algorithm 4.3 with $P_1 - P_1 - P_2$. 151
4.24 Error functions for Algorithm 4.4 with $P_1 - P_1 - P_2$. 151
4.25 Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_1$ 152
4.26 Error functions for Algorithm 4.3 with $P_2 - P_1 - P_1$. 153
4.27 Error functions for Algorithm 4.4 with $P_2 - P_1 - P_1$. 153
4.28 Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_2$. 154
4.29 Error functions for Algorithm 4.3 with $P_2 - P_1 - P_2$. 155
4.30 Error functions for Algorithm 4.4 with $P_2 - P_1 - P_2$. 155
4.31 Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_3$. 156
4.32 Error functions of Algorithm 4.3 with $P_2 - P_1 - P_3$ 157
4.33 Error functions for Algorithm 4.4 with $P_2 - P_1 - P_3$. 157
4.34 Mesh Analysis of Algorithm 4.5 with Spaces $P2 - P1 - P1$ . . . 163
4.35 Error Functions of Algorithm 4.5 with Spaces $P2 - P1 - P1$ . . 164

5.1 Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_1 - P_1$ Elements. 223

5.2 Error Functions for Gauge-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963). 224
5.3 Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_1 - P_1$ Elements. 225
5.4 Error Functions for Gauge-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963). 226
5.5 Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_2 - P_1$ Elements. 227
5.6 Error Functions for Gauge-Uzawa Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371). 228
5.7 Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements. 229
5.8 Error Functions for Gauge-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371). 230
5.9 Error Decay of Gauge-Uzawa with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2. 232
5.10 Error Functions of Gauge-Uzawa with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963). . . . 233
5.11 Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements. 234
5.12 Numerical Solution of Gauge-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903). 235
5.13 Initial and Steady State Solutions. 236
5.14 Driven Cavity Flow for $h = \frac{1}{128}$, $\triangle t = 1$, $Re = 10,000$. 237

6.1 Initial and Boundary Values of Thermal Driven Cavity 277

6.2 Thermal Driven Cavity of Experiment 1 at time=0.003. 278
6.3 Thermal Driven Cavity of Experiment 1 at time=0.01. 278
6.4 Thermal Driven Cavity of Experiment 1 at time=0.025. 279
6.5 Thermal Driven Cavity of Experiment 1 at time=0.1. 279
6.6 Thermal Driven Cavity of Experiment 1 at time=0.2. 280
6.7 Thermal Driven Cavity of Experiment 2 at time=30. 281
6.8 Thermal Driven Cavity of Experiment 2 at time=100. 282
6.9 Thermal Driven Cavity of Experiment 2 at time=250. 282
6.10 Thermal Driven Cavity of Experiment 2 at time=1000. 283
6.11 Thermal Driven Cavity of Experiment 2 at time=2000. 283
6.12 Initial and Boundary Values of Benard Example 284
6.13 The Benard Example at $t = 0.05$ 285
6.14 The Benard Example at $t = 0.10$ 286
6.15 The Benard Example at $t = 0.15$ 287
6.16 Steady State Solution of Benard Example $t = 1.0$ 288

Chapter 1

Introduction

1.1 Navier-Stokes Equations

The mathematical description of viscous fluid flows is given by the Navier-Stokes equations (NSE), a system of partial differential equations. The following mathematical model for physically relevant incompressible flows in a bounded domain Ω in $\mathbb{R}^d$ $(d = 2 \text{ or } 3)$ is the starting point for the subsequent numerical algorithms,

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \dfrac{1}{Re} \triangle \mathbf{u} = \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \tag{1.1.1}$$

The unknowns $(\mathbf{u}, p)$ are the velocity vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and the scalar pressure field $p = p(\mathbf{x}, t)$. Function $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is the given forcing term, $Re > 0$ is the Reynolds number, and $\mathbf{u}_0$ is a given initial velocity.

Equations (1.1.1) are difficult to solve numerically for several reasons. The first difficulty arises from the incompressibility constraint, which requires a discrete inf-sup condition between finite elements spaces for velocity and pressure [12]. The second is the lack of initial and boundary values of pressure which leads to inconsistencies and boundary layer formation for projection type methods. The third difficulty is due to stability which limits the applicability of many schemes to low Reynolds numbers. Finally, a theoretical and essential difficulty is the regularity of the exact solution, which may not be smooth enough to get optimal convergence order.

The mixed finite element method is a classical and effective algorithm to solve the stationary Stokes equations, and it has been applied to (1.1.1) upon coupling with suitable time discretizations. It leads to a saddle point formulation with relatively higher computational cost than the Poisson equation due to the size and indefinite nature of the resulting matrix. In the late 60's, projection methods which decouple velocity and pressure were introduced independently by Chorin [4] and Temam [26, 27], and several other projection type methods were constructed later to avoid the difficulties of previous methods [2, 18, 21]. Projection type methods consist of two main computational steps per time iteration:

(a) Momentum step : find an artificial velocity $\widehat{\mathbf{u}}$ which satisfies an approximate momentum equation but not the incompressible constraint.

(b) Projection step : decompose the artificial velocity $\widehat{\mathbf{u}}$ into a divergence free velocity $\mathbf{u}$ and a gradient field.

Some methods include one or two other artificial variables to improve accuracy. Most projection methods exhibit inconsistencies and require initial or boundary value of pressure which are not physical. This leads to the formation of boundary layers for pressure and a related loss of accuracy. Chapter 2 introduces some projection methods, displays inconsistencies along with their artificial boundary conditions. Finally numerical and theoretical results are compared with those of Gauge-Uzawa method.

E and Liu introduced the gauge method to overcome these disadvantages of projection type methods [7]. The gauge method employs the gauge variable ϕ and the auxiliary field $\mathbf{a} = \mathbf{u} - \nabla\phi$, which in turn lead to two gauge formulations (3.1.4) and (3.1.6) equivalent to (1.1.1). The main advantage of these gauge formulations is to enforce boundary conditions on the non-physical variable ϕ, which is smoother than p, without degrading the approximation of p. In Chapter 3, we construct 4 time-discrete gauge methods by combining formulations (3.1.4) and (3.1.6), with two Neumann and Dirichlet boundary conditions (3.1.7) and (3.1.8) for ϕ. We prove energy error estimates under realistic regularity assumptions via a variational approach. This improves upon the error analysis of Liu and Wang [30], which relies on asymptotic expansions, requires strong regularity only valid for special flows, and yields no estimate for pressure.

The chief disadvantage of these 4 methods is the implementation of such boundary conditions at the discrete level, because they involve both normal and tangential derivatives of ϕ. We introduce variational methods to perform accurate boundary computations of normal and tangential derivatives of ϕ in the discrete finite element spaces in Chapter 4. This restricts the applicability of gauge methods to 2d. Since the boundary conditions of $\mathbf{a}^{n+1}$ and ϕ^{n+1} are coupled, an explicit boundary condition has to be imposed on $\mathbf{a}^{n+1}$. This extrapolation technique causes an incompatibility for the Dirichlet condition $\phi = 0$, which has not been noticed earlier. Another serious difficulty is to find compatible finite element spaces, because of the unusual couplings of the gauge variable ϕ with the auxiliary field $\mathbf{a}$ through boundary values and with the pressure p via the heat equation. So ϕ demands a higher polynomial degree than one for pressure and velocity for a stable computation, which is not optimal in terms of degrees of freedom since ϕ is just an auxiliary variable.

In order to discover and study stable finite element spaces, we apply gauge methods to the stationary Stokes system in Section 4.2. Numerical experiments with many different combinations of finite element space show that the error due to decoupling boundary conditions is bigger than the inf-sup stability error. This can be avoided if ϕ has a higher degree than velocity and pressure, but at the expense of a larger computational cost. An important discovery is the equivalence between Gauge and Uzawa methods at the continuous level as iterative solvers for Stokes. In contrast to the Gauge method, which transfers data through the boundary, the Uzawa method does it by updating the pressure in the domain. In order to overcome the differentiation of ϕ on the boundary, we introduce the Gauge-Uzawa scheme: a Gauge method which transfers data as the Uzawa method.

In Section 4.4, we prove convergence of the Gauge-Uzawa method for the stationary Stokes equations via a variational approach, and find the convergence rate $1 - \beta^2$, where $\beta \leq 1$ is the inf-sup constant. This proof follows from the apparently new estimate $\|\text{div}\, \mathbf{u}\| \leq \|\nabla \mathbf{u}\|$ for all $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, and property $\beta \leq 1$. Numerical results for the Gauge-Uzawa method show much better results than for the Gauge method when applied to the stationary Stokes equations. Furthermore we discover that the Gauge-Uzawa method is just the Uzawa method with a divergence free velocity and a relaxation parameter $\alpha = 1$. This implies, in particular, that the relaxation parameter $\alpha > 0$ in the Uzawa method can be taken $\alpha = 1$, as was generally believed, instead of sufficiently small.

In Chapter 5, we introduce the Gauge-Uzawa (GU) method for the evolution Navier Stokes equation (1.1.1). This method exhibits several advantages with respect to other projection methods:

- GU leads naturally to a fully discrete algorithm, which is amenable to a complete and rigorous analysis. In contrast many projection-type methods are formulated as suitable time discretizations but are difficult to study once space is also discretized. Often, the discrete spaces for velocity, pressure, and other artificial variables such as ϕ, cannot be chosen arbitrarily. In contrast, GU does not incorporate inconsistencies or incompatibilities between space and time discretizations.

- In contrast to the Gauge method, GU does not require any boundary calculation and thus applies to any space dimension.

- GU entails a relatively low computational cost. Since ϕ is included in the pressure space, which is of lower polynomial degree than the velocity space, GU is more efficient than the Gauge method.

- GU is unconditionally stable, which extends its applicability to large Reynolds numbers.

We prove error estimates for both velocity and pressure under realistic regularity assumptions via a variational approach, extending the ideas and methodology in Chapter 3 to the fully discrete case. All our numerical experiments show that errors of GU are smaller than those of any other projection-type schemes.

In Chapter 6, we apply and analyze the GU method for the Boussinesq equations. In order to prove convergence under realistic regularity assumptions, we study the regularity of the exact solution of Boussinesq equations in Section 6.2.

We prove stability in Section 6.3 and derive error estimates in Sections 6.4 and 6.5 in the spirit of Chapter 5. We present the thermal driven cavity simulation in Section 6.6, which is a benchmark computation by Gresho et al. [13], and a stationary numerical solution of the Benard problem.

1.2 Preliminaries and Assumptions

This chapter is mainly devoted stating assumptions, reviewing some well-known lemmas, and to proving basic properties of (1.1.1). The basic mathematical theory of this section can be found in the work of Heywood and Rannacher [16], and in the books of A. Prohl [22] and Constantin and Foias [5].

Let $H^s(\Omega)$ be the Sobolev space of $L^2(\Omega)$-functions with s weak derivatives in $L^2(\Omega)$, $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$ and $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, where d is the space dimension. We will use $\|\cdot\|_0$ to denote the $L^2(\Omega)$-norm, and $\langle \cdot\,,\,\cdot\rangle$ to denote the $L^2(\Omega)$-inner product for both scalar and vector-valued functions. Let $\|\cdot\|_s$ denote the Sobolev norm of $\mathbf{H}^s(\Omega)$.

Consider the following Stokes equations:

$$\begin{cases} -\triangle \mathbf{v} + \nabla q = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0, & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

Assumption 1 (About the domain) *The unique solution* $(\mathbf{v}, q)$ *of the steady Stokes equation (1.2.1) satisfies*

$$\|\mathbf{v}\|_2 + \|q\|_1 \le C\|\mathbf{f}\|_0.$$

We remark that the validity of Assumption 1 is known if $\partial\Omega$ is of class $\mathbf{C}^2$ [5], or if $\partial\Omega$ is a two-dimensional convex polygon [17], and is generally believed for convex polyhedra [16].

Assumption 2 (Data regularity) *The velocity* $\mathbf{u}$ *and the forcing term in (1.1.1) satisfy*

$$\mathbf{u}(0) \in \mathbf{H}^2(\Omega) \cap \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\},$$

$$\mathbf{f},\ \mathbf{f}_t \in \mathbf{L}^\infty(0, \infty; \mathbf{L}^2(\Omega)).$$

Assumption 3 (Regularity of the solution $\mathbf{u}$) *There exists* $M \in \mathbb{R}$ *such that*

$$\sup_{t\in[0,T]} \|\nabla \mathbf{u}(t)\|_0 \leq M.$$

We note that Assumption 3 is automatically satisfied if $d = 2$ [16].

Let us introduce the following space which includes the solution $\mathbf{v}$ of the Stokes system (1.2.1):

$$\mathbf{Z}(\Omega) = \{\mathbf{z} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{z} = 0\}. \tag{1.2.2}$$

Then the space $\mathbf{Z}(\Omega)$ is a closed subspace of $\mathbf{H}_0^1(\Omega)$ [12]. And we denote by $\mathbf{Z}^*$ the dual space of $\mathbf{Z}(\Omega)$ normed by

$$\|\mathbf{f}\|_{\mathbf{Z}^*} = \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\langle \mathbf{f}\,,\, \mathbf{z}\rangle}{\|\mathbf{z}\|_1}. \tag{1.2.3}$$

Then we have the following lemma:

Lemma 1.1 (Norm equivalence) *Let* $(\mathbf{v}, q, \mathbf{f})$ *be the functions in the Stokes system (1.2.1). Then there exist two positive constants* C_1, C_2 *such that*

$$C_1\|\mathbf{f}\|_{\mathbf{Z}^*} \leq \|\mathbf{v}\|_1 \leq C_2\|\mathbf{f}\|_{\mathbf{Z}^*}.$$

PROOF.

$$\|\mathbf{f}\|_{\mathbf{Z}^*} = \sup_{\mathbf{z}\in\mathbf{Z}(\Omega)} \frac{\langle \mathbf{f}\,,\,\mathbf{z}\rangle}{\|\mathbf{z}\|_1} = \sup_{\mathbf{z}\in\mathbf{Z}(\Omega)} \frac{\langle \nabla\mathbf{v}\,,\,\nabla\mathbf{z}\rangle - \langle q\,,\,\mathrm{div}\,\mathbf{z}\rangle}{\|\mathbf{z}\|_1} \leq C\|\mathbf{v}\|_1.$$

And since $\mathbf{v} \in \mathbf{Z}$,

$$\|\mathbf{f}\|_{\mathbf{Z}^*} = \sup_{\mathbf{z}\in\mathbf{Z}(\Omega)} \frac{\langle \mathbf{f}\,,\,\mathbf{z}\rangle}{\|\mathbf{z}\|_1} \geq \frac{\langle -\triangle\mathbf{v} + \nabla q\,,\,\mathbf{v}\rangle}{\|\mathbf{v}\|_1} \geq C\|\mathbf{v}\|_1. \quad \blacksquare$$

We recall now the well known Sobolev Imbedding Theorem [11, 12]:

Lemma 1.2 *(Sobolev Imbedding Theorem) Let Ω be a bounded domain in $\mathbb{R}^d$, and let $\mathbf{u} \in \mathbf{W}_0^{1,p}(\Omega)$ be the space of $\mathbf{L}^p(\Omega)$ functions with weak derivatives in $\mathbf{L}^p(\Omega)$ and vanishing trace. Then there exists a constant $C = C(d,p)$ such that*

$$\|\mathbf{u}\|_{\mathbf{L}^{\frac{dp}{d-p}}(\Omega)} \leq C\|\nabla\mathbf{u}\|_{\mathbf{L}^p(\Omega)}, \quad \textit{for } p < d. \tag{1.2.4}$$

To handle the convection term in (1.1.1) it is convenient to introduce the trilinear form

$$\mathcal{N}(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}. \tag{1.2.5}$$

Then we have well known lemma [12]:

Lemma 1.3 *(Properties of $\mathcal{N}$) Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be in $\mathbf{H}^1(\Omega)$ and $\mathrm{div}\,\mathbf{u} = 0$. If*

$$\mathbf{u}\cdot\nu = 0 \quad \textit{or} \quad \mathbf{v} = 0 \quad \textit{on } \partial\Omega,$$

then

$$\mathcal{N}(\mathbf{u},\mathbf{v},\mathbf{w}) = -\mathcal{N}(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \textit{and} \quad \mathcal{N}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0.$$

Sobolev imbedding Lemma 1.2 yields the following results, which will be used later in dealing with the convection term of (1.1.1)

Lemma 1.4 *If $d \leq 4$, then*

$$\int_\Omega \mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}d\mathbf{x} \leq \begin{cases} C\|\mathbf{u}\|_0\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \\ C\|\mathbf{u}\|_2\|\mathbf{v}\|_0\|\mathbf{w}\|_0, \end{cases} \tag{1.2.6}$$

and if $d \leq 3$, then

$$\int_\Omega \mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}d\mathbf{x} \leq \begin{cases} C\|\mathbf{u}\|_1\|\mathbf{v}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{w}\|_0 \\ C\|\mathbf{u}\|_1\|\mathbf{v}\|_0^{\frac{1}{2}}\|\mathbf{v}\|_1^{\frac{1}{2}}\|\mathbf{w}\|_0. \end{cases} \tag{1.2.7}$$

Proof. Since (1.2.6) is proved in [12], we simply sketch the proof of (1.2.7). By Hölder and Young's inequalities, we get

$$\begin{aligned} \int_\Omega \mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}d\mathbf{x} &\leq C\|\mathbf{u}\cdot\mathbf{v}\|_0\|\mathbf{w}\|_0 \\ &\leq C\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{w}\|_0 \\ &\leq C\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{v}\|_0^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}^{\frac{1}{2}}\|\mathbf{w}\|_0. \\ &\leq C\|\mathbf{u}\|_1\|\mathbf{v}\|_0^{\frac{1}{2}}\|\mathbf{v}\|_1^{\frac{1}{2}}\|\mathbf{w}\|_0. \quad \blacksquare \end{aligned} \tag{1.2.8}$$

Heywood and Rannacher [16] proved the following:

Lemma 1.5 (A priori estimates) *Let the weight function $\sigma(t)$ be defined by*

$$\sigma(t) = \min\{t, 1\}. \tag{1.2.9}$$

Suppose Assumptions 1-3 hold, and let $0 < T \leq \infty$. Then there exists a constant $M > 0$ such that the solution of (1.1.1) satisfies

$$\sup_{0<t<T}\{\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\|_0 + \|p(t)\|_1\} \leq M, \tag{1.2.10}$$

$$\int_0^T \|\mathbf{u}_t(t)\|_1^2 dt \leq M, \quad \sup_{0<t<T}\sigma(t)\|\mathbf{u}_t(t)\|_1^2 \leq M, \tag{1.2.11}$$

and

$$\int_0^T \sigma(t)\{\|\mathbf{u}_t(t)\|_2^2 + \|\mathbf{u}_{tt}(t)\|_0^2 + \|p_t(t)\|_1^2\}dt \leq M. \qquad (1.2.12)$$

The following two lemmas indicate the condition to get rid of $\sigma(t)$:

Lemma 1.6 *Suppose Assumptions 1-3 hold and let* $0 < T \leq \infty$. *Then*

$$\|\nabla \mathbf{u}_t(0)\|_0 \leq C \qquad (1.2.13)$$

if and only if

$$\int_0^T \|\mathbf{u}_{tt}(t)\|_0^2 dt + \sup_{0<t<T} \|\nabla \mathbf{u}_t(t)\|_0^2 \leq M. \qquad (1.2.14)$$

Furthermore, if (1.2.13) holds, then

$$\int_0^T \left(\|p_t(t)\|_1^2 + \|\mathbf{u}_t(t)\|_2^2\right) dt \leq M. \qquad (1.2.15)$$

Proof. By differentiation of the momentum equation with respect to t, we get

$$\mathbf{u}_{tt} + (\mathbf{u}_t \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_t + \nabla p_t - \frac{1}{Re}\triangle \mathbf{u}_t = \mathbf{f}_t. \qquad (1.2.16)$$

First, we assume (1.2.13). Multiplying (1.2.16) by $\mathbf{u}_{tt}$ yields

$$\|\mathbf{u}_{tt}\|_0^2 + \frac{1}{Re}\langle \nabla \mathbf{u}_t\, , \, \nabla \mathbf{u}_{tt}\rangle = -\mathcal{N}(\mathbf{u}_t, \mathbf{u}, \mathbf{u}_{tt}) - \mathcal{N}(\mathbf{u}, \mathbf{u}_t, \mathbf{u}_{tt}) + \langle \mathbf{f}_t\, , \, \mathbf{u}_{tt}\rangle . \qquad (1.2.17)$$

Lemmas 1.4 and 1.5 lead to

$$\begin{aligned} \|\mathbf{u}_{tt}\|_0^2 + \frac{1}{2Re}\frac{d}{dt}\|\nabla \mathbf{u}_t\|_0^2 &\leq C\|\mathbf{u}_t\|_1\|\mathbf{u}\|_2\|\mathbf{u}_{tt}\|_0 + \|\mathbf{f}_t\|_0\|\mathbf{u}_{tt}\|_0 \\ &\leq C\|\mathbf{u}_t\|_1^2 + C\|\mathbf{f}_t\|_0^2 + \frac{1}{2}\|\mathbf{u}_{tt}\|_0^2. \end{aligned} \qquad (1.2.18)$$

On integrating in time t from 0 to T implies

$$\begin{aligned} \frac{1}{2}\int_0^T \|\mathbf{u}_{tt}\|_0^2 dt \quad &+ \quad \frac{1}{2Re}\|\nabla \mathbf{u}_t(T)\|_0^2 \\ &\leq \quad \frac{1}{2Re}\|\nabla \mathbf{u}_t(0)\|_0^2 + C\int_0^T \left(\|\mathbf{u}_t\|_1^2 + \|\mathbf{f}_t\|_0^2\right) dt. \end{aligned} \qquad (1.2.19)$$

By assumption (1.2.13) and Lemma 1.5, we derive (1.2.14). Since $t \mapsto \|\nabla \mathbf{u}_t(t)\|_0$ is continuous, (1.2.13) is a trivial consequence of (1.2.14).

To prove (1.2.15), we use the Helmholtz decomposition of $-\triangle \mathbf{u} \in \mathbf{L}^2(\Omega)$ as follows:

$$-\triangle \mathbf{u} = \mathbf{v} + \nabla q \tag{1.2.20}$$

with div $\mathbf{v} = 0$ and $\mathbf{v} \cdot \nu = 0$ on $\partial\Omega$. Since $(\mathbf{u}, -q)$ is a solution of the stationary Stokes problem with right-hand side $\mathbf{v}$, from Assumption 1 we have

$$\|\mathbf{u}\|_2 \leq C\|\mathbf{v}\|_0. \tag{1.2.21}$$

Multiplying (1.2.16) by $\mathbf{v}_t$ we arrive at

$$\langle \mathbf{u}_{tt}\, ,\, \mathbf{v}_t\rangle - \frac{1}{Re}\langle \triangle \mathbf{u}_t\, ,\, \mathbf{v}_t\rangle + \mathcal{N}(\mathbf{u}_t, \mathbf{u}, \mathbf{v}_t) + \mathcal{N}(\mathbf{u}, \mathbf{u}_t, \mathbf{v}_t) = \langle \mathbf{f}_t\, ,\, \mathbf{v}_t\rangle\, . \tag{1.2.22}$$

Since $\mathbf{u} = 0$ on $\partial\Omega$ and div $\mathbf{u} = 0$ in Ω, we can write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_t\|_0^2 + \frac{1}{Re}\|\mathbf{v}_t\|_0^2 &\leq C\|\mathbf{u}_t\|_1\|\mathbf{u}\|_2\|\mathbf{v}_t\|_0 + C\|\mathbf{f}_t\|_0\|\mathbf{v}_t\|_0 \\ &\leq CRe\|\mathbf{u}_t\|_1^2 + CRe\|\mathbf{f}_t\|_0^2 + \frac{1}{2Re}\|\mathbf{v}_t\|_0. \end{aligned} \tag{1.2.23}$$

Integrating in time t from 0 to T, we deduce

$$\begin{aligned} \|\nabla \mathbf{u}_t(T)\|_0^2 + \frac{1}{Re}\int_0^T \|\mathbf{v}_t\|_0^2 dt &\leq \|\nabla \mathbf{u}_t(0)\|_0^2 \\ &\quad + CRe\int_0^T \left(\|\mathbf{u}_t\|_1^2 + \|\mathbf{f}_t\|_0^2\right) dt. \end{aligned} \tag{1.2.24}$$

In view of (1.2.13) and $\|\mathbf{u}_t\|_2 \leq C\|\mathbf{v}_t\|_0$, we infer that

$$\int_0^T \|\mathbf{u}_t\|_2^2 dt \leq C. \tag{1.2.25}$$

Now we prove

$$\int_0^T \|p_t\|_1^2 dt \leq C. \tag{1.2.26}$$

From (1.2.16)

$$\begin{aligned}\|\nabla p_t\|_0 &\le \|\mathbf{u}_{tt}\|_0 + \|\mathbf{u}_t\|_0\|\mathbf{u}\|_1 + \|\mathbf{u}\|_0\|\mathbf{u}_t\|_1 + \frac{1}{Re}\|\triangle \mathbf{u}_t\|_0 + \|\mathbf{f}_t\|_0 \\ &\le C\left(\|\mathbf{u}_{tt}\|_0 + \|\mathbf{u}_t\|_0 + \|\mathbf{u}_t\|_1 + \frac{1}{Re}\|\mathbf{u}_t\|_2 + \|\mathbf{f}_t\|_0\right).\end{aligned} \tag{1.2.27}$$

Squaring and integrating in time from 0 to T, and using (1.2.13), and (1.2.25), we easily get (1.2.26) and finish the proof. ■

Lemma 1.7 *Suppose Assumptions 1-3 hold, and let* $0 < T \le \infty$. *Then we have*

$$\int_0^T \|\mathbf{u}_{tt}(t)\|^2_{\mathbf{Z}^*} dt \le M. \tag{1.2.28}$$

Furthermore, if (1.2.13) hold, then

$$\sup_{0<t<T} \|\mathbf{u}_{tt}(t)\|^2_{\mathbf{Z}^*} \le M. \tag{1.2.29}$$

PROOF. Since $\|\nabla p\|_{\mathbf{Z}^*} = 0$, for all $\nabla p \in \mathbf{L}^2(\Omega)$, from (1.2.16) we get

$$\begin{aligned}\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*} &\le \|(\mathbf{u}_t\cdot\nabla)\mathbf{u}\|_{\mathbf{Z}^*} + \|(\mathbf{u}\cdot\nabla)\mathbf{u}_t\|_{\mathbf{Z}^*} + \frac{1}{Re}\|\triangle \mathbf{u}_t\|_{\mathbf{Z}^*} + \|\mathbf{f}_t\|_{\mathbf{Z}^*} \\ &= A_1 + A_2 + A_3 + A_4.\end{aligned} \tag{1.2.30}$$

Invoking Lemma 1.4, the convection term A_1 can be bounded by

$$\begin{aligned}A_1 &= \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\langle(\mathbf{u}_t\cdot\nabla)\mathbf{u}\,,\,\mathbf{z}\rangle}{\|\mathbf{z}\|_1} \\ &\le \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\|\mathbf{u}_t\|_0\|\mathbf{u}\|_2\|\mathbf{z}\|_1}{\|\mathbf{z}\|_1} \le C\|\mathbf{u}_t\|_0\|\mathbf{u}\|_2,\end{aligned} \tag{1.2.31}$$

and using Lemma 1.3, the other convection term A_2 becomes

$$\begin{aligned}A_2 &= \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\langle(\mathbf{u}\cdot\nabla)\mathbf{z}\,,\,\mathbf{u}_t\rangle}{\|\mathbf{z}\|_1} \\ &\le \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\|\mathbf{u}\|_2\|\mathbf{u}_t\|_0\|\nabla\mathbf{z}\|_0}{\|\mathbf{z}\|_1} \le C\|\mathbf{u}\|_2\|\mathbf{u}_t\|_0.\end{aligned} \tag{1.2.32}$$

The diffusion term A_3 can be bounded as follows:

$$A_3 = \frac{1}{Re} \sup_{\substack{\mathbf{z}\in\mathbf{Z}(\Omega)\\ \mathbf{z}\neq 0}} \frac{\langle \nabla \mathbf{u}_t\,,\,\nabla \mathbf{z}\rangle}{\|\mathbf{z}\|_1} \leq \frac{C}{Re}\|\nabla \mathbf{u}_t\|_0. \tag{1.2.33}$$

Inserting (1.2.31)-(1.2.33) back into (1.2.30), we get

$$\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*} \leq C\left(\|\mathbf{u}\|_2\|\mathbf{u}_t\|_0 + \|\mathbf{f}_t\|_0\right) + \frac{C}{Re}\|\nabla \mathbf{u}_t\|_0. \tag{1.2.34}$$

In view of Lemmas 1.5-1.6 and Assumption 2, integration in time yields (1.2.28) and (1.2.29). ■

Remark 1.8 Since Lemmas 1.6 and 1.7 are used to estimate only pressure error, we use (1.2.13) as an assumption in only pressure analysis sections.

Lemma 1.9 *Let* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$. *Then*

$$\|\text{div }\mathbf{u}\|_0 \leq \|\nabla \mathbf{u}\|_0. \tag{1.2.35}$$

Proof. We prove this in 2 dimensions, since it can be extended to higher dimensions without any additional effort. Given $\mathbf{u} = (u, v) \in \mathbf{H}_0^1(\Omega)$, there exists a sequence $\{\mathbf{u}^n\} \in \mathbf{C}_0^\infty(\Omega)$ such that

$$\|\text{div }(\mathbf{u}^n - \mathbf{u})\|_0 \to 0 \quad \text{and} \quad \|\nabla(\mathbf{u}^n - \mathbf{u})\|_0 \to 0, \quad \text{as} \quad n \to \infty. \tag{1.2.36}$$

Integrating by parts for $\mathbf{u}^n = (u^n, v^n) \in \mathbf{C}_0^\infty(\Omega)$ implies

$$\begin{aligned}
\|\text{div }\mathbf{u}^n\|_0^2 &= \int_\Omega (\partial_x u^n + \partial_y v^n)^2\, d\mathbf{x} \\
&= \int_\Omega \left((\partial_x u^n)^2 + 2\partial_x u^n \partial_y v^n + (\partial_y v^n)^2\right) d\mathbf{x} \\
&= \int_\Omega \left((\partial_x u^n)^2 + 2\partial_x v^n \partial_y u^n + (\partial_y v^n)^2\right) d\mathbf{x} \\
&\leq \int_\Omega \left((\partial_x u^n)^2 + (\partial_x v^n)^2 + (\partial_y u^n)^2 + (\partial_y v^n)^2\right) d\mathbf{x} \\
&= \|\nabla \mathbf{u}^n\|_0^2.
\end{aligned} \tag{1.2.37}$$

We thus have $\|\text{div } \mathbf{u}^n\|_0^2 \leq \|\nabla \mathbf{u}^n\|_0^2$, for all n. The assertion (1.2.35) follows from (1.2.36) by passing to the limit $n \to \infty$. ■

Lemma 1.9 is a simple, but apparently new, result with important implications (see Lemma 1.11 and main theorems). We now consider a finite element discretization. Let $\mathfrak{T} = \{K\}$ be a finite decomposition of meshsize h, $0 < h < 1$, of the polyhedral domain $\bar{\Omega}$ into closed elements K. The finite element spaces for velocity and pressure are of the form

$$\begin{aligned} \mathbb{V}_h &= \{\mathbf{v}_h \in \mathbf{H}^1(\Omega) : \mathbf{v}_h|_K \in \mathcal{V}(K), \quad \forall K \in \mathfrak{T}\} \\ \mathbb{P}_h &= \{p_h \in C^0(\Omega) : p_h|_K \in \mathcal{P}(K), \quad \forall K \in \mathfrak{T}, \quad \int_\Omega p_h d\mathbf{x} = 0\}, \end{aligned} \tag{1.2.38}$$

where $\mathcal{V}(K)$ and $\mathcal{P}(K)$ are spaces of uniformly bounded degree polynomials with respect to $K \in \mathfrak{T}$. We also define $\mathbb{V}_h^0$

$$\mathbb{V}_h^0 = \{\mathbf{v}_h \in \mathbb{V}_h : \mathbf{H}_0^1(\Omega)\}. \tag{1.2.39}$$

Remark 1.10 ($\mathbb{P}_h \subset C^0$ needed for some algorithms) The discrete velocity $\mathbf{u}_h$ for the Gauge-Uzawa method of Chapters 3, 4, and 5 is the sum of some $\widehat{\mathbf{u}}_h \in \mathbb{V}_h$ and $\rho_h \in \mathbb{P}_h$:

$$\mathbf{u}_h = \widehat{\mathbf{u}}_h + \nabla \rho_h. \tag{1.2.40}$$

So $\mathbf{u}_h$ is a discontinuous function across interelement boundaries, i.e. $\mathbf{u}_h \notin \mathbf{H}_0^1(\Omega)$. We impose a discrete divergence free constraint in the sense that

$$\langle \mathbf{u}_h \, , \, \nabla q_h \rangle = 0 \quad \text{for all} \quad q_h \in \mathbb{P}_h. \tag{1.2.41}$$

Hence,

$$\langle \nabla \rho_h \, , \, \nabla q_h \rangle = - \langle \widehat{\mathbf{u}}_h \, , \, \nabla q_h \rangle = \langle \text{div } \widehat{\mathbf{u}}_h \, , \, q_h \rangle \, , \quad \text{for all} \quad q_h \in \mathbb{P}_h. \tag{1.2.42}$$

This explain why discrete pressures must be continuous, a property used in most results below. However, this continuity is not necessary in the Uzawa method, Section 4.6, because the velocity in the Uzawa Algorithm 4.7 is not discrete divergence free.

The spaces $\mathbb{V}_h$ and $\mathbb{P}_h$ must satisfy the celebrated compatibility condition [12]:

Assumption 4 *(*Discrete Inf-Sup*) There exists a constant $\beta > 0$ such that*

$$\inf_{p_h \in \mathbb{P}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{\langle \operatorname{div} \mathbf{v}_h \, , \, p_h \rangle}{\|\mathbf{v}_h\|_1 \|p_h\|_0} \geq \beta. \tag{1.2.43}$$

An alternative way of writing (1.2.43) is

$$\begin{cases} \text{For all } p_h \in \mathbb{P}_h, \text{ there exists } \mathbf{v}_h \in \mathbb{V}_h \text{ such that} \\ \langle \operatorname{div} \mathbf{v}_h \, , \, p_h \rangle = \|p_h\|_0^2 \text{ and } \|\mathbf{v}_h\|_1 \leq \dfrac{1}{\beta} \|p_h\|_0. \end{cases} \tag{1.2.44}$$

Assumption 5 *(*Mesh shape regularity*) There exists a constant $C > 0$ such that for all $K \in \mathfrak{T}$*

$$\operatorname{diam}(B_K) \geq C \operatorname{diam}(K), \tag{1.2.45}$$

where B_K is the largest ball contained in K.

Assumption 6 *(*Approximability*) Let assume $\mathbf{v} \in \mathbf{H}^{s+1}$ and $q \in H^s$. And let $m+1$ and m be a polynomial degrees of $\mathbb{V}_h$ and $\mathbb{P}_h$, respectively. If we define*

$$\kappa = \min\{s, m\}, \tag{1.2.46}$$

then there exist approximations $\mathbf{v}_h \in \mathbb{V}_h$ and $q_h \in \mathbb{P}_h$ such that

$$\begin{aligned} \|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 &\leq Ch^{\kappa+1} \|\mathbf{v}\|_{\kappa+1}, \\ \|q - q_h\|_0 &\leq Ch^{\kappa} \|q\|_{\kappa}. \end{aligned} \tag{1.2.47}$$

Assumption 7 *(Discrete Initial Condition) The discrete initial value $\mathbf{u}_h^0$ is the $\mathbf{L}^2$-projection of $\mathbf{u}_0$ into $\mathbb{V}_h$:*

$$\langle \mathbf{u}_h^0 \,,\, \mathbf{w}_h \rangle = \langle \mathbf{u}_0 \,,\, \mathbf{w}_h \rangle \,, \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \tag{1.2.48}$$

By Lemma 1.9, we can find an upper bound of the inf-sup constant β in Assumption 4.

Lemma 1.11 *Let β be the constant in the inf-sup condition (1.2.43). Then we have*

$$\beta \leq 1. \tag{1.2.49}$$

PROOF. Let p_h be in the pressure space $\mathbb{P}_h$. By the Assumption 4, there exists $\mathbf{v}_h \in \mathbb{V}_h$ such that

$$\langle \mathrm{div}\, \mathbf{v}_h \,,\, p_h \rangle = \|p_h\|_0^2 \quad \text{and} \quad \|\mathbf{v}_h\|_1 \leq \frac{1}{\beta}\|p_h\|_0. \tag{1.2.50}$$

Then, by (1.2.50) and Lemma 1.9, we get

$$\begin{aligned} \|p_h\|_0^2 &= \langle \mathrm{div}\, \mathbf{v}_h \,,\, p_h \rangle \\ &\leq \|\mathrm{div}\, \mathbf{v}_h\|_0 \|p_h\|_0 \\ &\leq \|\nabla \mathbf{v}_h\|_0 \|p_h\|_0 \leq \frac{1}{\beta}\|p_h\|_0^2. \end{aligned} \tag{1.2.51}$$

So we get $\beta \leq 1$. ■

We now recall some useful results. We start with the well known inverse inequality lemma [11, 12]:

Lemma 1.12 *(Inverse Inequality) Let the mesh $\mathfrak{T}$ satisfy Assumption 5. Let $\mathbf{u}_h \in \mathbb{V}_h(\Omega)$, and $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 \leq m \leq n$. Then*

$$\|\mathbf{u}_h\|_{\mathbf{W}_p^n(K)} \leq C h^{m-n+\frac{d}{p}-\frac{d}{q}} \|\mathbf{u}_h\|_{\mathbf{W}_q^m(K)}. \tag{1.2.52}$$

Lemma 1.13 *(A Priori Energy Estimate) The unique solution* $(\mathbf{v}, q)$ *of the steady state Stokes equations (1.2.1) satisfies*

$$\|\mathbf{v}\|_1 + \|q\|_0 \leq C\|\mathbf{f}\|_{-1}. \tag{1.2.53}$$

Now, we define the finite element formulation of Stokes equations (1.2.1) : find $\mathbf{v}_h \in \mathbb{V}_h$ and $q_h \in \mathbb{P}_h$ such that

$$\begin{cases} \langle \nabla \mathbf{v}_h \,,\, \nabla \mathbf{w}_h \rangle - \langle q_h \,,\, \mathrm{div}\, \mathbf{w}_h \rangle = \langle \mathbf{f} \,,\, \mathbf{w}_h \rangle \,, & \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle r_h \,,\, \mathrm{div}\, \mathbf{v}_h \rangle = 0, & \forall r_h \in \mathbb{P}_h. \end{cases} \tag{1.2.54}$$

Lemma 1.14 *Let* $(\mathbf{v}, q)$ *and* $(\mathbf{v}_h, q_h)$ *be solutions of the Stokes equations (1.2.1) and (1.2.54), respectively. If the Assumptions 1-6 hold, then*

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 + h\|q - q_h\|_0 \leq Ch^2 \left(\|\mathbf{v}\|_2 + \|q\|_1\right). \tag{1.2.55}$$

Lemma 1.15 *Let* $(\mathbf{v}_h, q_h) \in \mathbb{V}_h \times \mathbb{P}_h$ *be a discrete solution of the steady state Stokes equations (1.2.54). If Assumption 4 is valid, the we have*

$$\|\mathbf{v}_h\|_1 + \|q_h\|_0 \leq C\|\mathbf{f}\|_{-1}. \tag{1.2.56}$$

Proof. By choosing $\mathbf{w}_h = \mathbf{v}_h$ in (1.2.54), we have

$$\|\mathbf{v}_h\|_1 \leq C\|\mathbf{f}\|_{-1}. \tag{1.2.57}$$

Since $q_h \in \mathbb{P}_h$, Assumption 4 implies the existence of $\mathbf{w}_h \in \mathbb{V}_h$ such that

$$\langle \mathrm{div}\, \mathbf{w}_h \,,\, q_h \rangle = \|q_h\|_0^2 \quad \text{and} \quad \|\mathbf{w}_h\|_1 \leq \frac{1}{\beta}\|q_h\|_0. \tag{1.2.58}$$

So, by using (1.2.57) and (1.2.58),

$$\begin{aligned}\|q_h\|_0^2 &= \langle \operatorname{div} \mathbf{w}_h\,,\, q_h\rangle \\ &= \langle \nabla \mathbf{v}_h\,,\, \nabla \mathbf{w}_h\rangle - \langle \mathbf{f}\,,\, \mathbf{w}_h\rangle \\ &\le \|\nabla \mathbf{v}_h\|_0 \|\nabla \mathbf{w}_h\|_0 + C\|\mathbf{f}\|_{-1}\|\mathbf{w}_h\|_1 \\ &\le \frac{C}{\beta}(\|\mathbf{f}\|_{-1} + \|\nabla \mathbf{v}_h\|_0)\|q_h\|_0 \le \frac{C}{\beta}\|\mathbf{f}\|_{-1}\|q_h\|_0.\end{aligned} \tag{1.2.59}$$

We thus deduce the assertion (1.2.56). ■

The discrete analogue $\mathcal{N}_h$ of the form $\mathcal{N}(\mathbf{u},\mathbf{v},\mathbf{w}) = \langle (\mathbf{u}\cdot\nabla)\mathbf{v}\,,\,\mathbf{w}\rangle$ of (1.2.5) is defined by

$$\mathcal{N}_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = \frac{1}{2}\left\langle (\mathbf{u}_h\cdot\nabla)\mathbf{v}_h\,,\,\mathbf{w}_h\right\rangle - \frac{1}{2}\left\langle (\mathbf{u}_h\cdot\nabla)\mathbf{w}_h\,,\,\mathbf{v}_h\right\rangle. \tag{1.2.60}$$

This definition implies that $\mathcal{N}_h$ is skew-symmetric, that is

$$\mathcal{N}_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = -\mathcal{N}_h(\mathbf{u}_h,\mathbf{w}_h,\mathbf{v}_h) \quad \text{and} \quad \mathcal{N}_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{v}_h) = 0. \tag{1.2.61}$$

The following lemma can be derived from Lemma 1.2 and Hölder inequality [16, 25]:

Lemma 1.16 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be in* $\mathbf{H}^2(\Omega)$ *with* $\operatorname{div}\mathbf{u} = 0$*, and let* $\mathbf{u}_h$*,* $\mathbf{v}_h$*,* $\mathbf{w}_h \in \mathbb{V}_h$*. Then*

$$\mathcal{N}_h(\mathbf{u},\mathbf{v}_h,\mathbf{w}_h) \le \begin{cases} C\|\mathbf{u}\|_2\|\nabla\mathbf{v}_h\|_0\|\mathbf{w}_h\|_0 \\ C\|\mathbf{u}\|_2\|\mathbf{v}_h\|_0\|\nabla\mathbf{w}_h\|_0, \end{cases} \tag{1.2.62}$$

$$\mathcal{N}_h(\mathbf{u}_h,\mathbf{v},\mathbf{w}_h) \le C\|\mathbf{u}_h\|_0\|\mathbf{v}\|_2\|\nabla\mathbf{w}_h\|_0. \tag{1.2.63}$$

If $d \le 3$*,*

$$\mathcal{N}_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) \le \begin{cases} C\|\mathbf{u}_h\|_0 |\!|\!|\mathbf{v}_h|\!|\!| \|\nabla\mathbf{w}_h\|_0 \\ C\|\mathbf{u}_h\|_{\mathbf{L}^3(\Omega)}\|\nabla\mathbf{v}_h\|_0\|\nabla\mathbf{w}_h\|_0, \end{cases} \tag{1.2.64}$$

where $|\!|\!|\mathbf{v}_h|\!|\!| = \|\mathbf{v}_h\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla\mathbf{v}_h\|_{\mathbf{L}^3(\Omega)}$.

In light of Lemma 1.12, we can write the inverse inequality

$$|||\mathbf{v}_h||| \leq Ch^{-\frac{d}{2}}\|\mathbf{v}_h\|_0 + Ch^{-\frac{d}{6}}\|\mathbf{v}_h\|_1. \tag{1.2.65}$$

We derive thus the following

Lemma 1.17 *Let* $\mathbf{v} \in \mathbf{H}^2(\Omega)$ *and let* $\mathbf{v}_h \in \mathbb{V}_h$ *satisfy*

$$\|\mathbf{v} - \mathbf{v}_h\|_0 \leq Ch^{\frac{d}{2}} \quad \textit{and} \quad \|\mathbf{v} - \mathbf{v}_h\|_1 \leq Ch^{\frac{d}{6}}. \tag{1.2.66}$$

Then

$$|||\mathbf{v} - \mathbf{v}_h||| \leq M. \tag{1.2.67}$$

1.3 Benchmark Problems

In order to compare the different algorithms that will be dealt with in the subsequent chapters, we consider the following model problems. All numerical experiments are calculated with the finite element toolbox ALBERT of A. Schmidt and K. Siebert [23]. All graphics are produced by MATLAB and a graphic tollbox GRAPE [24].

1.3.1 Example: Smooth Solution

The computational domain is $\Omega = [0, 1] \times [0, 1]$ and Re is chosen to be 1. We choose the following exact solution of (1.1.1) and determine the corresponding force term $\mathbf{f}(t)$

$$\begin{cases} u(x, y, t) = \cos(t)(x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3) \\ v(x, y, t) = -\cos(t)(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3) \\ p(x, y, t) = \cos(t)\left(x^2 + y^2 - \dfrac{2}{3}\right). \end{cases} \tag{1.3.1}$$

In all experiments of Chapter 4 which are time independent, we choose the exact solution (1.3.1) with $t = 0$. In order to avoid cancellations due to mesh uniformity and symmetry, we choose the distorted macromesh of Figure 1.1 (a) which is further refined uniformly via bisection. Two elementary bisection operations are shown in Figure 1.1 (b). Diagrams (a) and (b) in Figure 1.2 depict quasi-uniform and regular meshes after 4 levels of refinement. Table 1.1 gives the relation between refinement and finite element structure. Mesh distortion is crucial to uncover numerical difficulties that may go unnoticed otherwise. For instance, Gauge method is insensitive to the discrete inf-sup condition for uniform mesh (b) in Figure 1.2. In order to check the dependency of inf-sup condition on symmetric and equidistance mesh, we make a mesh analysis on the uniform mesh (b) in Figure 1.2 with linear polynomial degree of velocity and pressure which does not satisfy inf-sup condition. In order to compare numerical behavior of several schemes, we choose the following 4 combinations of discretization parameters and polynomial degrees (K_v = polynomial degree of velocity, K_p = polynomial degree of pressure):

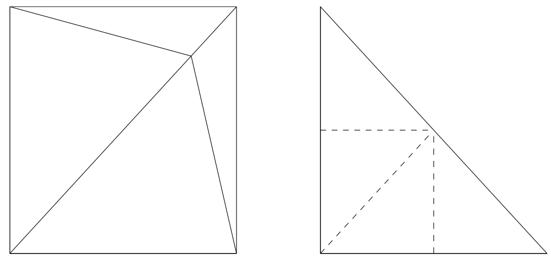
Case 1 : $\triangle t = h^2$, $K_v = 2$, $K_p = 1$.

Case 2 : $\triangle t = h^2$, $K_v = K_p = 1$.

Case 3 : $\triangle t = h$, $K_v = 2$, $K_p = 1$.

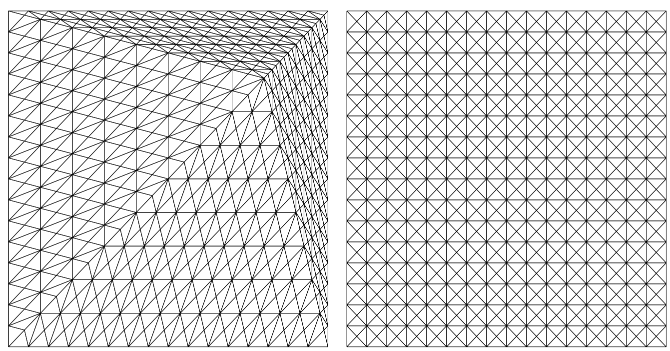
Case 4 : $\triangle t = h$, $K_v = K_p = 1$.

Each algorithm shows different dependence on these combinations. As we know, the finite element spaces of Cases 1 and 3 correspond to the Taylor-Hood family $P_2 - P_1$ which satisfies the discrete inf-sup condition. In contrast, the finite element pairs $P_1 - P_1$ of Case 2 and 4 do not satisfy the discrete inf-sup condition.



(a) (b)

Figure 1.1: (a) Macro mesh, (b) Refinement of a triangle



(a) Quasi-Uniform Mesh (b) Regular Mesh

Figure 1.2: Quasi-Uniform and Regular Meshes with 4 Levels of Refinement

Since the truncation error can be split into space and time contributions, we compute with relations $\triangle t = h^2$ and $\triangle t = h$. If $\triangle t = h$ and the space errors are $\mathcal{O}(h^{\kappa+1})$ with $\kappa \geq 1$, then the time error $\mathcal{O}(\triangle t)$ dominates the calculation.

1.3.2 Example: Singular Solution

We consider the L-shaped domain

$$\Omega = ((-1,1) \times (-1,1)) - ((0,1] \times [-1,0])$$

Refine-ment	Vertexes	Triangles	Edges	h	Unknowns		
					P_1	P_2	P_3
3	145	256	400	1/8	145	545	1,201
4	545	1,024	1,568	1/16	545	2,113	4,705
5	2,113	4,096	6,208	1/32	2,113	8,321	18,625
6	8,321	16,384	24,704	1/64	8,321	33,025	74,113
7	33,025	65,536	98,560	1/128	33,025	131,585	295,681

Table 1.1: The Relation between Refinement Level and Finite Element Structure

with reentrant angle $\omega = \frac{3\pi}{2}$ at the origin. Let $\alpha = 0.544$ be an approximation of the smallest root of the nonlinear equation:

$$\frac{\sin^2(\alpha\omega) - \alpha^2 \sin^2(\omega)}{\alpha^2} = 0.$$

The exact velocity $\mathbf{u}$ and pressure p for the stationary Stokes equation are given in polar coordinates by [29]

$$\mathbf{u}(r,\theta) = \cos(t) r^{\alpha} \begin{bmatrix} \cos(\theta)\psi'(\theta) + (1+\alpha)\sin(\theta)\psi(\theta) \\ \sin(\theta)\psi'(\theta) - (1+\alpha)\cos(\theta)\psi(\theta) \end{bmatrix} \tag{1.3.2}$$

and

$$p(r,\theta) = -\cos(t) r^{\alpha-1} \frac{(1+\alpha)^2\psi'(\theta) + \psi'''(\theta)}{1-\alpha}, \tag{1.3.3}$$

where $\psi(\theta)$ is the function

$$\begin{aligned} \psi(\theta) \quad = \quad & \frac{\sin((1+\alpha)\theta)\cos(\alpha\omega)}{1+\alpha} - \cos((1+\alpha)\theta) \\ & + \frac{\sin((\alpha-1)\theta)\cos(\alpha\omega)}{\alpha-1} - \cos((\alpha-1)\theta). \end{aligned} \tag{1.3.4}$$

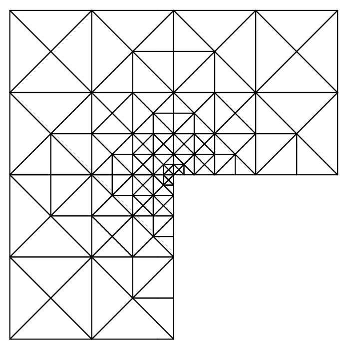


Figure 1.3: The Computational Mesh for Singular Solution

Chapter 2

Projection Methods

Many projection type methods have been constructed to solve NSE (1.1.1), but most of them include structural inconsistencies and artificial boundary layers. In this chapter, we introduce the original Chorin method and the Chorin-Uzawa method due to A. Prohl. Our main goal is to display their performance to be able to compare them with the Gauge-Uzawa method in Chapters 3 and 5.

2.1 Chorin Method

The original projection method was constructed by Chorin in 1968 [4, 8, 21]:

Algorithm 2.1 *(*Chorin Method*) Start with* $\mathbf{u}^0 = \mathbf{u}(0)$.

Step 1: (Momentum Equation) *Find* $\widetilde{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\widetilde{\mathbf{u}}^{n+1} - \dfrac{1}{Re}\triangle\widetilde{\mathbf{u}}^{n+1} = \mathbf{f}(t_{n+1}), & \text{in } \Omega, \\ \widetilde{\mathbf{u}}^{n+1} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

Step 2: (Projection step)

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\triangle t} + \nabla p^{n+1} = 0, & \textit{in } \Omega, \\ \text{div } \mathbf{u}^{n+1} = 0, & \textit{in } \Omega, \\ \mathbf{u}^{n+1} \cdot \nu = 0, & \textit{on } \partial\Omega. \end{cases} \tag{2.1.2}$$

In the projection step, we use the Helmholtz decomposition theorem to split $\widetilde{\mathbf{u}}^{n+1}$ into its solenoidal and irrotational parts $\mathbf{u}^{n+1}$ and ∇p^{n+1}. The divergence operator applied to (2.1.2) decouples the problem into two simple equations:

$$\begin{cases} \triangle p^{n+1} = \dfrac{1}{\triangle t} \text{div } \widetilde{\mathbf{u}}^{n+1} & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1.3}$$

and

$$\mathbf{u}^{n+1} = \widetilde{\mathbf{u}}^{n+1} - \triangle t \nabla p^{n+1}. \tag{2.1.4}$$

By virtue of $\mathbf{u}^{n+1} \cdot \nu = 0$, pressure p automatically satisfies the non-physical Neumann boundary condition $\dfrac{\partial p}{\partial \nu} = 0$. This artificial boundary condition is responsible for a non-physical boundary layer for p. Upon plugging (2.1.4) into (2.1.1) we also discover an inconsistency in the momentum equation

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\widetilde{\mathbf{u}}^{n+1} - \frac{1}{Re}\triangle \mathbf{u}^{n+1} + \nabla p^{n+1} - \frac{\triangle t}{Re}\triangle \nabla p^{n+1} = \mathbf{f}(t_{n+1}), \tag{2.1.5}$$

that is $-\dfrac{\triangle t}{Re}\triangle \nabla p^{n+1}$ is the inconsistent term. Notice the presence of the operator $-\triangle \nabla$, which will appear later on in our discussions.

There are several publications concerning error estimates for Chorin Algorithm 2.1. The most relevant for us is Prohl [21] who employs a variational approach. If $\sigma(t) = \min\{t, 1\}$ and Assumptions 1-3 hold, then [21]

$$\begin{aligned} \left\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right\|_0 + \sigma(t_{n+1})\left\|p(t_{n+1}) - p^{n+1}\right\|_{-1} &\le C\triangle t, \\ \left\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right\|_1 + \sqrt{\sigma(t_{n+1})}\left\|p(t_{n+1}) - p^{n+1}\right\|_0 &\le C\sqrt{\triangle t}. \end{aligned} \tag{2.1.6}$$

The second paper of interest [8] is by E and Liu, who derive error estimates via an asymptotic expansion approach: If the exact solution $(\mathbf{u}(t), p(t))$ of (1.1.1) is smooth, then

$$\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right\|_0 + \sqrt{\triangle t}\left\|p(t_{n+1}) - p^{n+1}\right\|_0 \le C\triangle t. \tag{2.1.7}$$

This result requires regularity which is often not valid for realistic incompressible flows.

2.2 Chorin-Uzawa Method

Trying to get rid of the boundary layer and inconsistency of Chorin method, had led to a number of papers [2, 18, 21]. We mention the following Chorin-Uzawa method of Prohl [21]:

Algorithm 2.2 *(*Chorin-Uzawa Method*) Start with given data* $(\mathbf{u}^0, p^0, \widetilde{p}^0)$ *such that*

$$\|\mathbf{u}(0) - \mathbf{u}^0\|_0 + \sqrt{\triangle t}\|p(0) - p^0\|_0 \le C\triangle t, \quad \widetilde{p}^0 = 0. \tag{2.2.1}$$

Step 1: (Momentum Equation) *Find* $\widetilde{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\widetilde{\mathbf{u}}^{n+1} - \dfrac{1}{Re}\triangle\widetilde{\mathbf{u}}^{n+1} + \nabla\left(p^n - \widetilde{p}^n\right) = \mathbf{f}(t_{n+1}), & \textit{in } \Omega, \\ \widetilde{\mathbf{u}}^{n+1} = 0, & \textit{on } \partial\Omega, \end{cases} \tag{2.2.2}$$

Step 2: (Projection Step)

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\triangle t} + \nabla \widetilde{p}^{n+1} = 0, & in\ \Omega, \\ \text{div } \mathbf{u}^{n+1} = 0, & on\ \Omega. \\ \mathbf{u}^{n+1} \cdot \nu = 0, & on\ \partial\Omega, \end{cases} \tag{2.2.3}$$

Step 3: (Pressure Step)

$$p^{n+1} = p^n - \frac{\alpha}{Re} \text{div } \widetilde{\mathbf{u}}^{n+1}, \quad 0 < \alpha < 1. \tag{2.2.4}$$

The Chorin-Uzawa method is a combination of Chorin method 2.1 and Uzawa Algorithm, an iterative solver of the stationary Stokes equations [1, 12]. The relaxation parameter α in (2.2.4) must be taken sufficiently small for the iteration to converge. Since we will prove convergence of Uzawa Algorithm for $\alpha = 1$ in Section 4.6, we can choose $\alpha = 1$ in (2.2.4).

Note the presence of the auxiliary pressure $\widetilde{p}^n$ with artificial boundary value $\dfrac{\partial \widetilde{p}^n}{\partial \nu} = 0$ in (2.2.3). No boundary condition is imposed on pressure p^n any longer. Regardless of this improvement, Chorin-Uzawa exhibits the following pitfalls:

- Initial Pressure: The initial value p^0 can not be chosen arbitrarily, because the initial error $\|p(0) - p^0\|$ enters in the priori error analysis.

- Inconsistency: Upon plugging $\widetilde{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} + \triangle t \nabla \widetilde{p}^{n+1}$ from (2.2.3) into (2.2.2), we see that

$$\begin{aligned} &\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\widetilde{\mathbf{u}}^{n+1} + \nabla \left(p^n - \frac{\alpha \triangle t}{Re} \triangle \widetilde{p}^{n+1} \right) \\ -&\frac{1}{Re} \triangle \mathbf{u}^{n+1} + (\alpha - 1) \frac{\triangle t}{Re} \nabla \triangle \widetilde{p}^{n+1} + \nabla \left(\widetilde{p}^{n+1} - \widetilde{p}^n \right) = \mathbf{f}(t_{n+1}). \end{aligned} \tag{2.2.5}$$

Since (2.2.3) and (2.2.4) imply $p^{n+1} = p^n - \dfrac{\alpha \triangle t}{Re} \triangle \widetilde{p}^{n+1}$, we end up with

$$\begin{aligned} &\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\widetilde{\mathbf{u}}^{n+1} + \nabla p^{n+1} - \frac{1}{Re} \triangle \mathbf{u}^{n+1} \\ &+ (\alpha - 1) \frac{\triangle t}{Re} \nabla \triangle \widetilde{p}^{n+1} + \nabla \left(\widetilde{p}^{n+1} - \widetilde{p}^{n}\right) = \mathbf{f}(t_{n+1}). \end{aligned} \tag{2.2.6}$$

Here $(\alpha - 1) \dfrac{\triangle t}{Re} \nabla \triangle \widetilde{p}^{n+1} + \nabla \left(\widetilde{p}^{n+1} - \widetilde{p}^{n}\right)$ is the inconsistency term. If we choose $\alpha = 1$ as in Section 4.6, then the first disappears but the second still remains.

- Regularity: The following regularity assumption of the exact solution $(\mathbf{u}, p)$ of (1.1.1) is used in the error analysis for both velocity and pressure of [21].

$$\sup_{0 \le s \le T} \left(\|\nabla \mathbf{u}_t(s)\|_0 + \|p_t(s)\|_0 \right) \le C. \tag{2.2.7}$$

The following a priori error bound is stated by Prohl [21]: If Assumptions 1-3 and (2.2.7) hold, then

$$\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right\|_1 + \sqrt{\triangle t}\left\|p(t_{n+1}) - p^{n+1}\right\|_0 \le C \triangle t. \tag{2.2.8}$$

We note that assumption (2.2.7) is required by the reason why the exact solution may not satisfy (1.2.13), $\|\nabla \mathbf{u}_t(0)\|_0 \le C$. Also Lemma 1.2.9 is satisfied automatically without $\sigma(t)$ under the assumption (1.2.13). Therefore we may say that assumption $\|\nabla \mathbf{u}_t(0)\|_0 \le C$ is equivalent with (2.2.7). As we said in Remark 1.8, we do not these assumptions to estimate velocity.

2.3 Numerical Results for Chorin and Chorin-Uzawa Methods

In this section we present a number of simulations using both the Chorin method and the Chorin-Uzawa method. Our aim is to compare they performance for both velocity and pressure in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^\infty(\Omega)$, and several combinations of finite element spaces.

2.3.1 Experiments with $P_1 - P_1$ Elements

We take polynomials of degree 1 for both velocity and pressure. This combination does not satisfy the discrete inf-sup condition. The errors of Chorin method decrease in Figures 2.1 and 2.4. This indicates that Chorin method is insensitive to the inf-sup condition. In contrast, the pressure error of Chorin-Uzawa in Figure 2.1 with $\triangle t = h^2$ does not converge to 0. So Chorin-Uzawa is sensitive to the inf-sup condition. A plausible explanation for the pressure error in Figure 2.4 is that the time discretization error dominates the space discretization error. The conclusion of these experiments is that the Chorin method seems to be insensitive to the discrete inf-sup condition, but that the Chorin-Uzawa method does not.

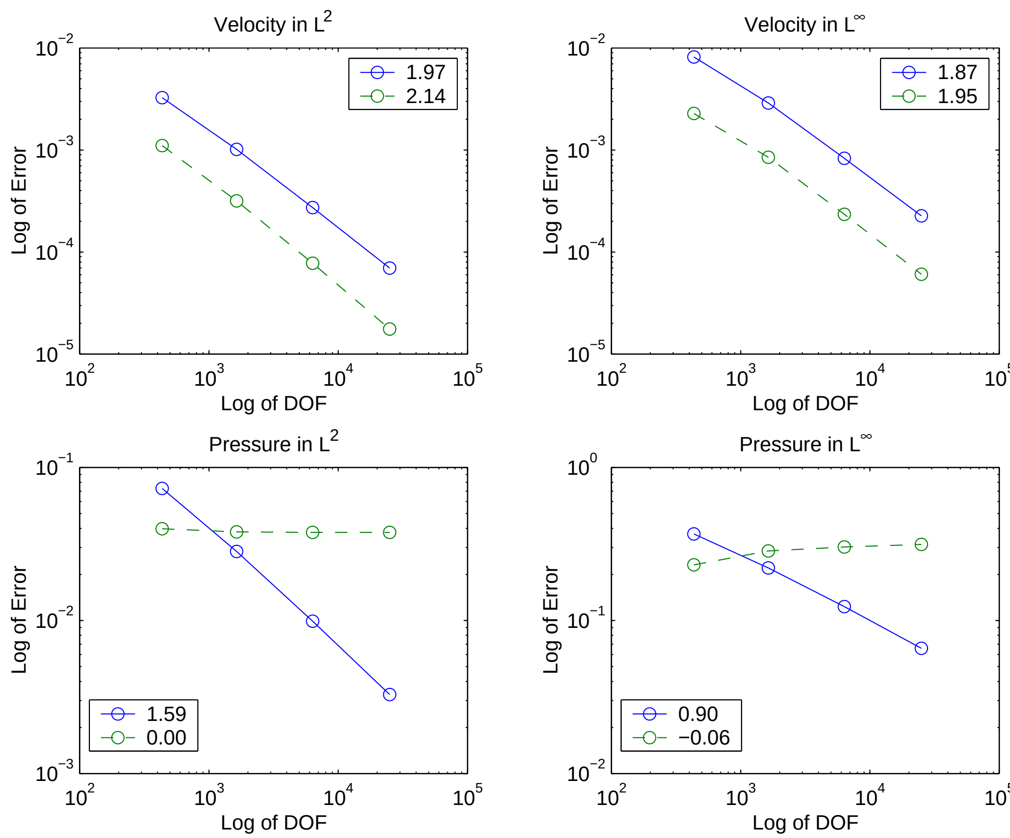


Figure 2.1: Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_1 - P_1$ Elements.

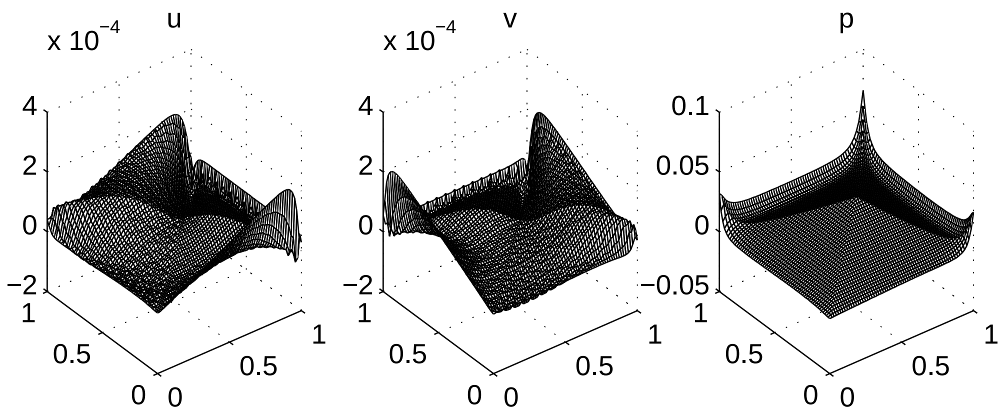


Figure 2.2: Error Functions for Chorin Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963).

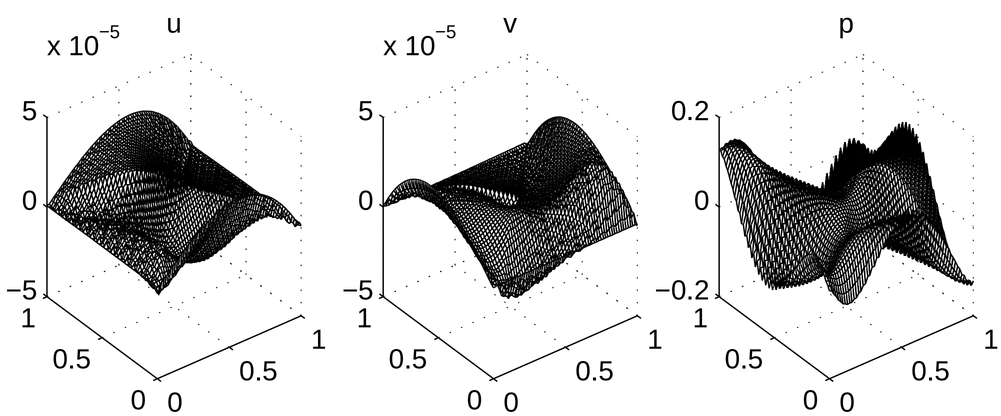


Figure 2.3: Error Functions for Chorin-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963).

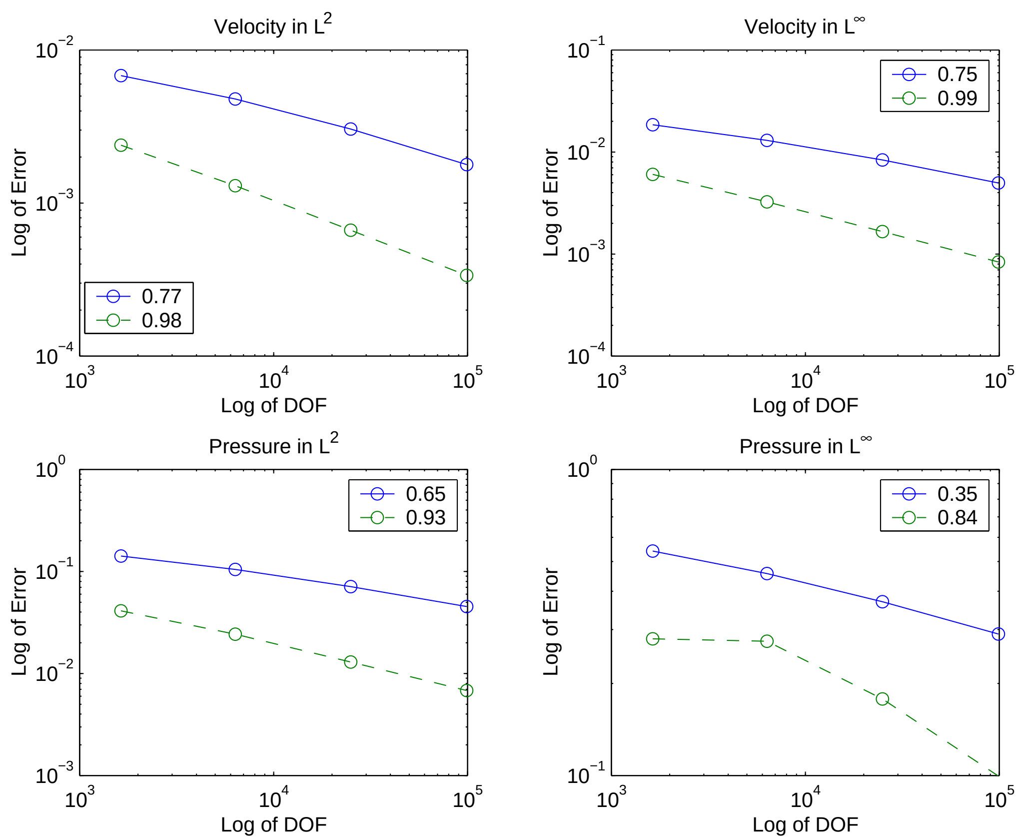


Figure 2.4: Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_1 - P_1$ Elements.

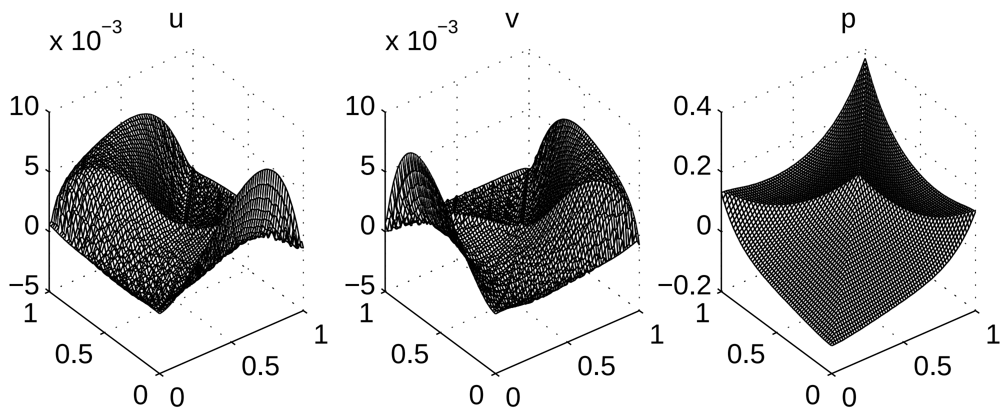


Figure 2.5: Error Functions for Chorin Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963).

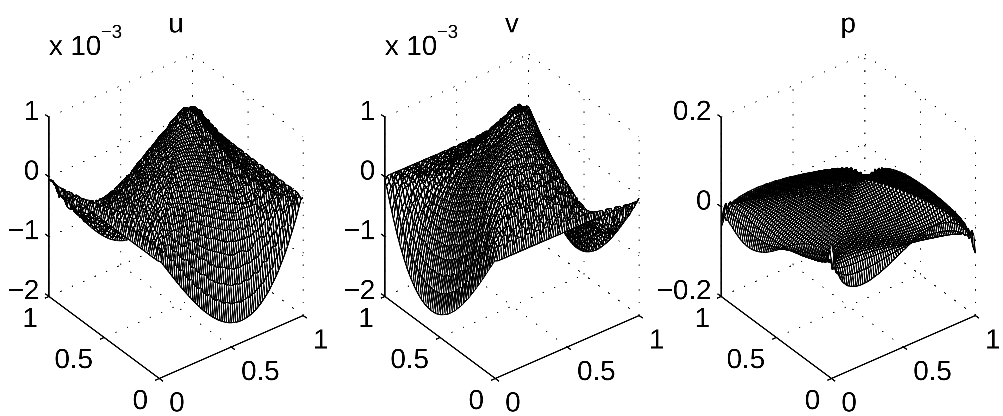


Figure 2.6: Error Functions for Chorin-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963).

2.3.2 Experiments with $P_2 - P_1$ Elements

We take polynomials of degree 2 for velocity and degree 1 for pressure. This is the so called Taylor-Hood family which satisfies the discrete inf-sup Assumption 4. Figure 2.7 shows the error decay for both methods and reveals that Chorin-Uzawa is more accurate. Figures 2.8 and 2.9 display the error functions for both methods. Figures 2.7-2.9 correspond to the relation $\triangle t = h^2$ whereas Figures 2.10-2.12 repeat the experiments with $\triangle t = h$.

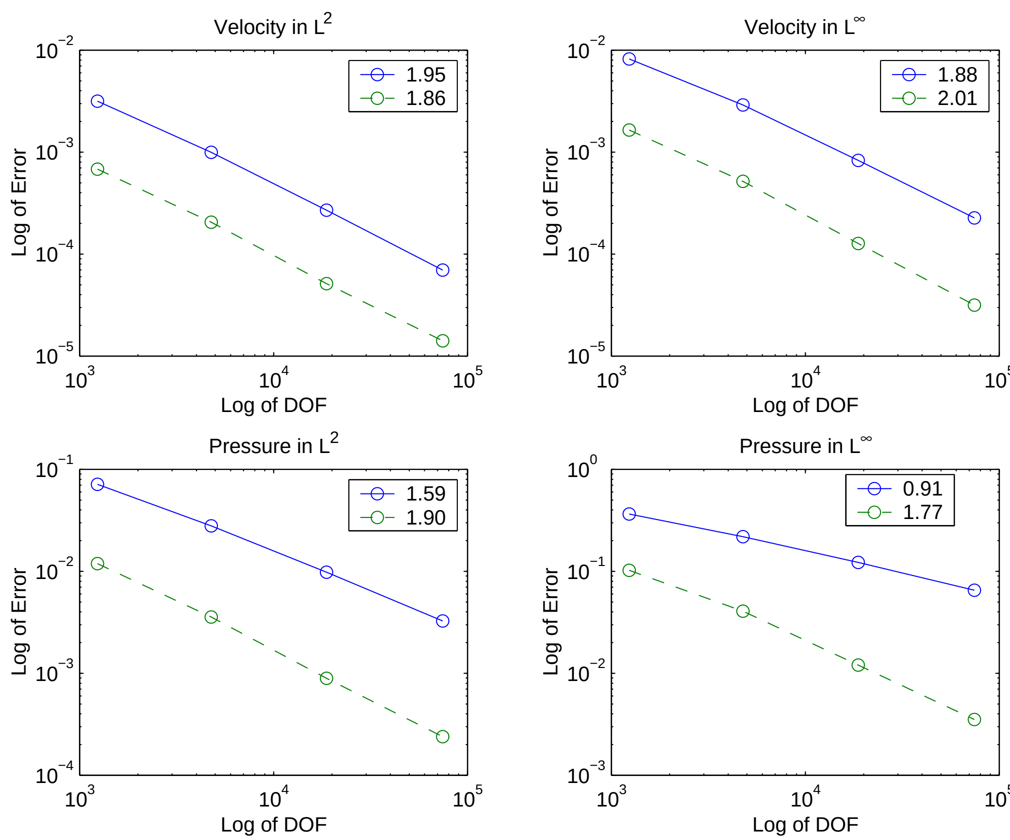


Figure 2.7: Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_2 - P_1$ Elements.

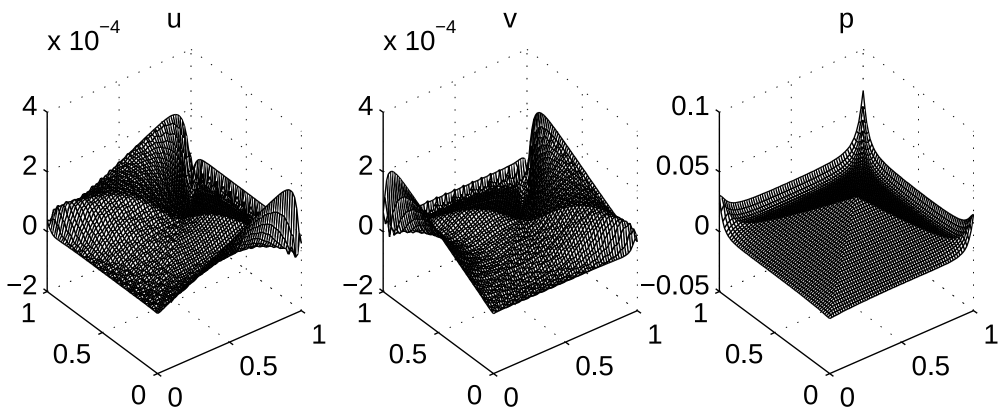


Figure 2.8: Error Functions for Chorin Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371).

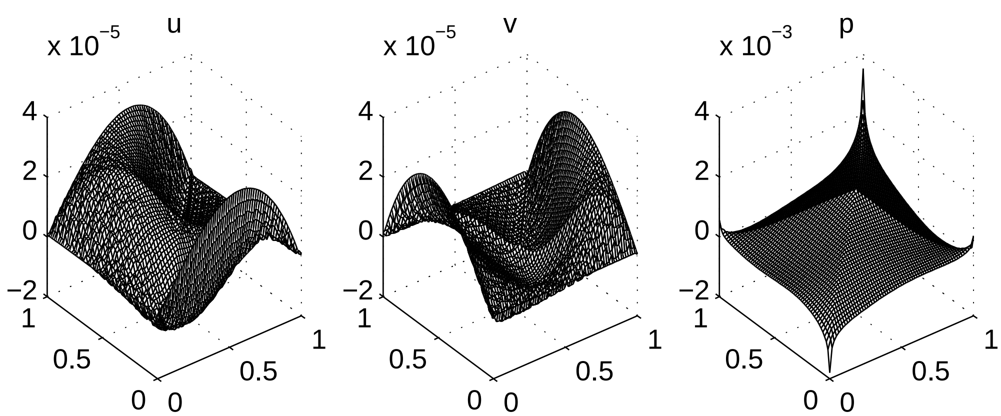


Figure 2.9: Error Functions for Chorin-Uzawa Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371).

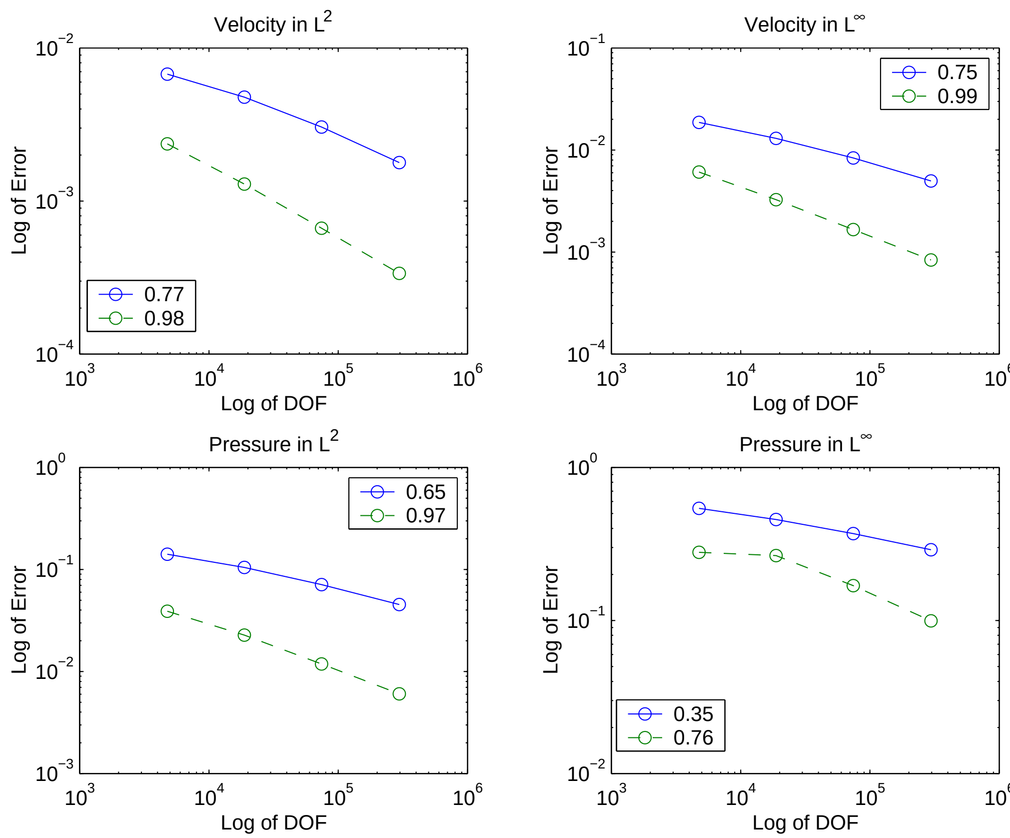


Figure 2.10: Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements.

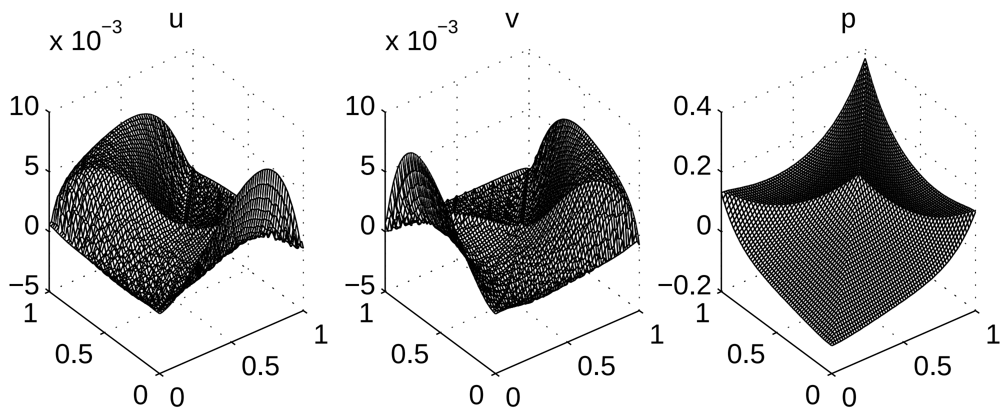


Figure 2.11: Error Functions for Chorin Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371).

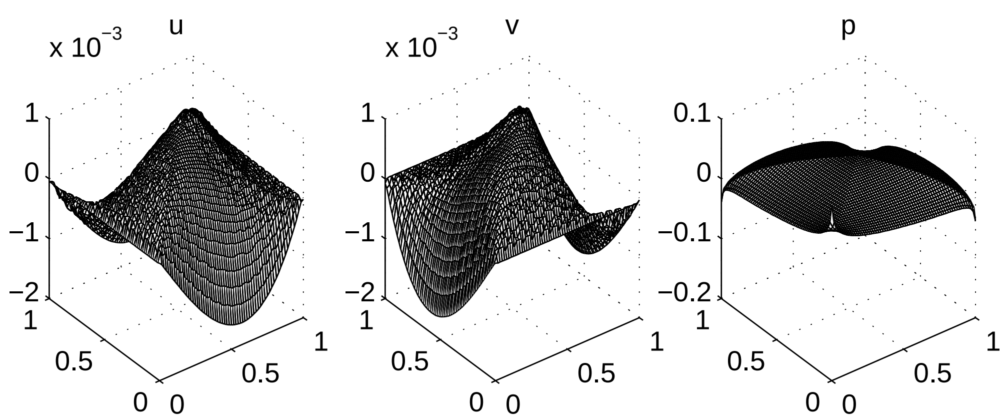


Figure 2.12: Error Functions for Chorin-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371).

2.3.3 Experiments for Chorin-Uzawa with $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2

The purpose of this experiment is to inspect whether the Chorin-Uzawa method depends on the inf-sup condition under regular mesh. The Figure 2.13 show that the error of pressure for Chorin-Uzawa method is not decreasing even on regular domain the same as Figure (2.1). Thus we conclude that Chorin-Uzawa method is sensitive to the discrete inf-sup condition. In contrast the gauge method in Chapter 3 does not depend on discrete inf-sup condition in the regular mesh (compare with Figure 3.25).

2.3.4 Example : Singular Solution

We perform the example 1.3.2 including a singular pressure p at the reentrant corner on domain Figure 1.3. We employ the Taylor-Hood finite element combination. The error of Chorin-Uzawa method is not uniformly decreasing in Figure 2.15 as decreasing mesh size, and it seems to relate with the pick of velocity at the reentrant corner in Figure 2.16. Even though the errors of Chorin method are decreasing at a uniform rate, the pressure at reentrant corner Figure 2.17 looks smooth in contrast with exact pressure. Comparing with the results of Gauge-Uzawa method in Figures 5.11 and 5.12 makes us recognize how the small inconsistencies 2.1.5 and 2.2.6 are crucial problem.

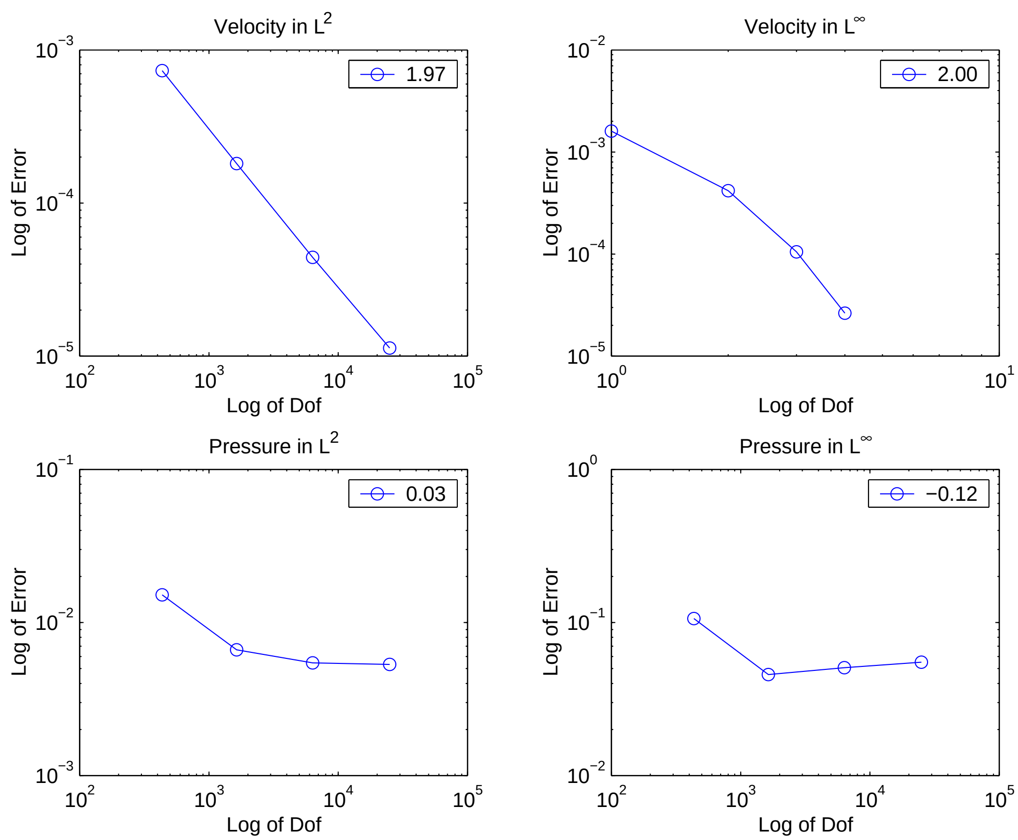


Figure 2.13: Error Decay of Chorin-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2.

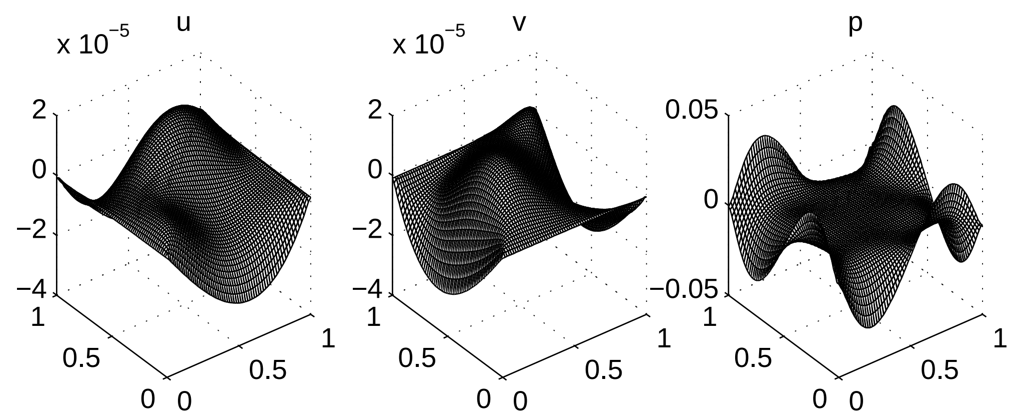


Figure 2.14: Error Functions of Chorin-Uzawa Method with $\triangle t = h^2$ Chorin-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963).

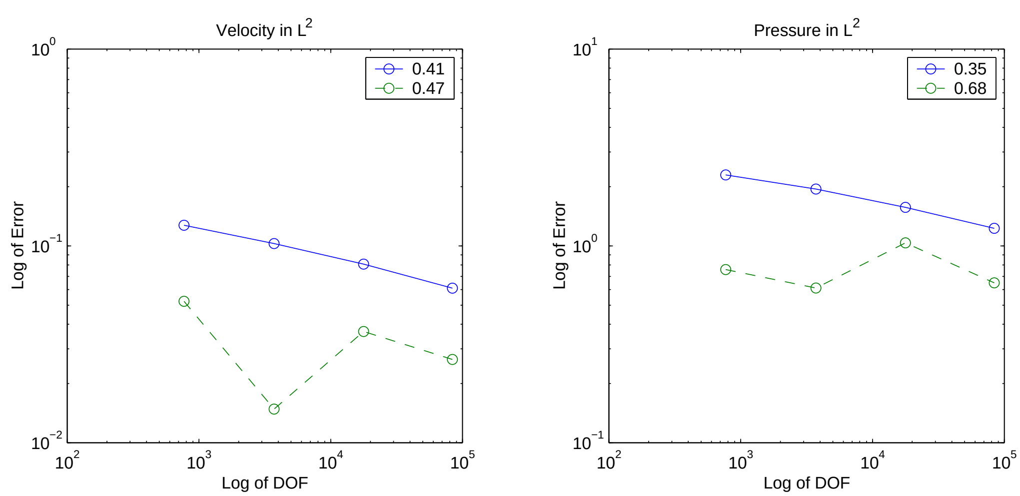


Figure 2.15: Error Decay of Chorin Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements.

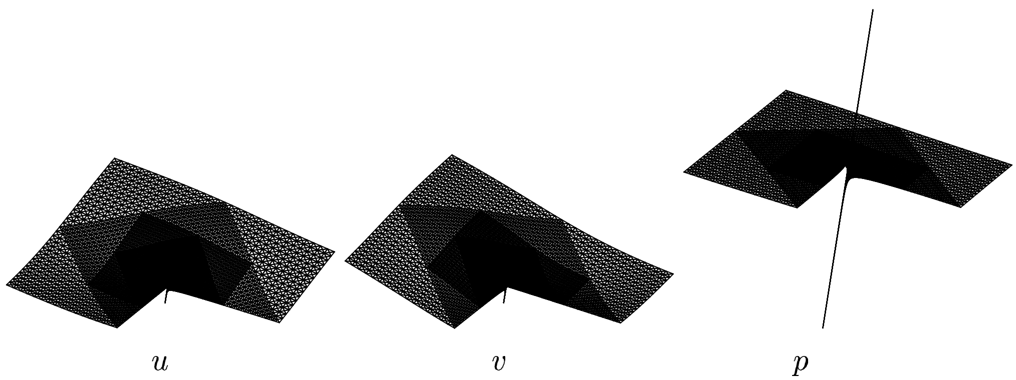


Figure 2.16: Numerical Solution of Chorin-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903).

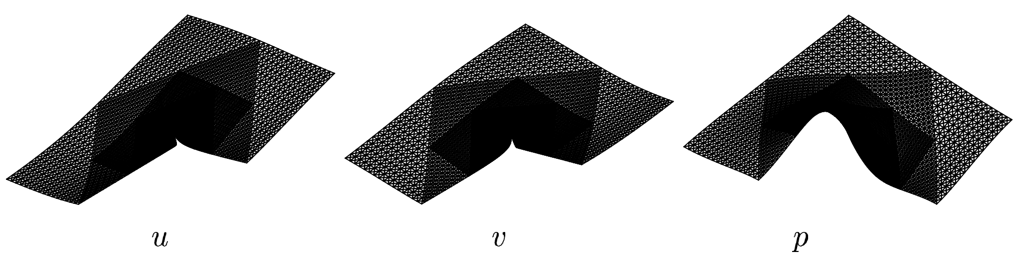


Figure 2.17: Numerical Solution of Chorin Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903).

Chapter 3

Gauge Method

E and Liu [7] proposed the gauge formulation of the incompressible Navier-Stokes equations along with the gauge method, which results from time discretization of the gauge formulation. In this chapter, we introduce 4 time-discrete gauge methods, study their properties, prove error estimates, and display some advantages and disadvantages of them. This study gives rise to the basic concept of Gauge-Uzawa method, which will be studied in Chapter 5.

3.1 Motivation of Gauge Method

The gauge formulation can be derived by introducing the gauge variable ϕ and the auxiliary field $\mathbf{a} = \mathbf{u} - \nabla\phi$. Then the momentum equation of (1.1.1) becomes

$$\mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\left(\phi_t - \frac{1}{Re}\triangle\phi + p\right) - \frac{1}{Re}\triangle\mathbf{a} = \mathbf{f}. \tag{3.1.1}$$

If we impose

$$\phi_t - \frac{1}{Re}\triangle\phi = -p, \tag{3.1.2}$$

we end up with the gauge formulation,

$$\begin{cases} \mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{Re}\triangle\mathbf{a} = \mathbf{f}, & \text{in } \Omega, \\ \mathbf{a} = \mathbf{u} - \nabla\phi, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \phi_t - \dfrac{1}{Re}\triangle\phi = -p, & \text{in } \Omega. \end{cases} \tag{3.1.3}$$

To derive an equation for ϕ, we apply the divergence operator to $\mathbf{a} = \mathbf{u} - \nabla\phi$. The incompressibility constraint $\operatorname{div}\mathbf{u} = 0$ in (3.1.3) becomes $-\triangle\phi = \operatorname{div}\mathbf{a}$. So (3.1.3) changes into the gauge formulation of NSE due to E and Liu [7], namely,

$$\begin{cases} \mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{Re}\triangle\mathbf{a} = \mathbf{f}, & \text{in } \Omega, \\ -\triangle\phi = \operatorname{div}\mathbf{a}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} + \nabla\phi, & \text{in } \Omega, \\ \phi_t - \dfrac{1}{Re}\triangle\phi = -p, & \text{in } \Omega. \end{cases} \tag{3.1.4}$$

In two space dimensions, we can also apply the rotation operator to both sides of $\mathbf{a} = \mathbf{u} - \nabla\phi$ to get $\operatorname{rot}\mathbf{a} = \operatorname{rot}\mathbf{u}$. Since $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{u} = 0$ on $\partial\Omega$, there exists an unique stream function ψ such that [12]

$$\begin{cases} \mathbf{curl}\,\psi = \mathbf{u}, & \text{in } \Omega, \\ \psi = C, \quad \dfrac{\partial\psi}{\partial\nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1.5}$$

The boundary value C in (3.1.5) can be chosen by simply 0 in connected domain. Finally, exploiting properties of rot and **curl** in (3.1.5), we get

$$-\triangle\psi = \operatorname{rot}\mathbf{curl}\,\psi = \operatorname{rot}\mathbf{u} = \operatorname{rot}\mathbf{a}.$$

Combining the above formulas with the gauge momentum equation, we can obtain a novel gauge formulation:

$$\begin{cases} \mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{Re}\triangle\mathbf{a} = \mathbf{f}, & \text{in } \Omega, \\ -\triangle\psi = \text{rot } \mathbf{a}, & \text{in } \Omega, \\ \psi = 0, \quad \dfrac{\partial\psi}{\partial\nu} = 0, & \text{on } \partial\Omega, \\ \mathbf{u} = \mathbf{curl}\,\psi, & \text{in } \Omega, \\ \nabla\phi = \mathbf{u} - \mathbf{a}, & \text{in } \Omega, \\ -p = \phi_t - \dfrac{1}{Re}\triangle\phi, & \text{in } \Omega. \end{cases} \tag{3.1.6}$$

The main advantage of these gauge formulations is the freedom of choice of boundary conditions for the non-physical gauge variable ϕ without degrading the approximation of p. This is reflected in the fact that ϕ is smoother than p, since ϕ is a solution of the heat equation with forcing term $-p$. To enforce the boundary condition $\mathbf{u} = 0$, we can either prescribe

$$\frac{\partial\phi}{\partial\nu} = 0, \qquad \mathbf{a}\cdot\nu = 0, \qquad \mathbf{a}\cdot\tau = -\frac{\partial\phi}{\partial\tau}, \tag{3.1.7}$$

or

$$\phi = 0, \qquad \mathbf{a}\cdot\nu = -\frac{\partial\phi}{\partial\nu}, \qquad \mathbf{a}\cdot\tau = 0, \tag{3.1.8}$$

where ν and τ are the unit vectors in the normal and tangential directions, respectively. We call (3.1.7) the Neumann formulation and (3.1.8) the Dirichlet formulation.

Wang and Liu show in [30] for the backward Euler time discretization of (3.1.4), that velocity is of order 1 for the Neumann formulation and of order $\frac{1}{2}$ for Dirichlet formulation. Since [30] is based on asymptotic analysis, the exact solutions are assumed to be sufficiently smooth, a rather strong and unrealistic regularity requirement. In addition, [30] does not address the convergence of pressure, which is the most sensitive variable. We use, instead, a variational technique to get optimal rates of convergence for both velocity and pressure under realistic regularity assumptions on data. A distinctive aspect of our study is the assessment of pressure convergence. Since pressure is obtained through differentiation of ϕ, the boundary conditions (3.1.7) and (3.1.8) play a central role. We explain in Section 3.2 that the Dirichlet condition (3.1.8) entails a rather strong compatibility condition on data for convergence of pressure. In contrast, the Neumann condition (3.1.7) always leads to pressure convergence; this issue is discussed in Section 3.6. Furthermore, we introduce a semi-implicit scheme and show that it is unconditionally stable in Section 3.3. This extends applicability of the gauge method to high Reynolds numbers.

3.2 Time Discretization and Algorithms

We consider the backward Euler time discretization of gauge methods (3.1.4) and (3.1.6) with both Neumann (3.1.7) and Dirichlet (3.1.8) conditions. In order to decouple the calculation of $\mathbf{a}^{n+1}$ and ϕ^{n+1} at time step $n+1$, it is necessary to extrapolate the boundary conditions from the previous time step. This extrapolation is responsible for a boundary layer, and a compatibility condition associated with the Dirichlet formulation. We first introduce two algorithms which use a Neumann boundary condition.

Algorithm 3.1 *(*Gauge Method (3.1.4) with Neumann Condition (3.1.7)*) Start with initial values* $\phi^0 = 0$ *and* $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$.

Step 1: *Find* $\mathbf{a}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{\mathbf{a}^{n+1}-\mathbf{a}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n - \dfrac{1}{Re}\triangle \mathbf{a}^{n+1} = \mathbf{f}(t_{n+1}), & \text{in } \Omega, \\ \mathbf{a}^{n+1} \cdot \nu = 0, \quad \mathbf{a}^{n+1} \cdot \tau = -\dfrac{\partial \phi^n}{\partial \tau}, & \text{on } \partial\Omega, \end{cases} \tag{3.2.1}$$

Step 2: *Find* ϕ^{n+1} *as the solution of*

$$\begin{cases} -\triangle \phi^{n+1} = \operatorname{div} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \dfrac{\partial \phi^{n+1}}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2.2}$$

Step 3: *Find*

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla \phi^{n+1}, \quad \text{in } \Omega. \tag{3.2.3}$$

Algorithm 3.2 *(*Gauge Method (3.1.6) with Neumann Condition (3.1.7)*) Start with initial values* $\phi^0 = 0$ *and* $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$.

Step 1: *Find* $\mathbf{a}^{n+1}$ *as the solution of (3.2.1)*

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{cases} -\triangle \psi^{n+1} = \operatorname{rot} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \psi^{n+1} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2.4}$$

Step 3: *Find*

$$\begin{aligned} \mathbf{u}^{n+1} &= \mathbf{curl}\, \psi^{n+1}, && \text{in } \Omega, \\ \nabla \phi^{n+1} &= \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, && \text{in } \Omega. \end{aligned} \tag{3.2.5}$$

One may compute the pressure whenever necessary as

$$p^{n+1} = -\frac{\phi^{n+1} - \phi^n}{\triangle t} + \frac{1}{Re}\triangle \phi^{n+1}. \tag{3.2.6}$$

Remark 3.1 (Boundary Condition) In Algorithms 3.1 and 3.2, the boundary condition of velocity $\mathbf{u}^{n+1}$ is

$$\mathbf{u}^{n+1} \cdot \nu = 0, \quad \mathbf{u}^{n+1} \cdot \tau = \frac{\partial \phi^{n+1}}{\partial \tau} - \frac{\partial \phi^n}{\partial \tau} \quad \text{on } \partial\Omega. \tag{3.2.7}$$

Note that $\mathbf{u}^{n+1} \cdot \tau$ is not zero because of the extrapolated boundary condition $\mathbf{a}^{n+1} \cdot \tau = -\frac{\partial \phi^n}{\partial \tau}$. In order to reduce the boundary layer in (3.2.7), we can use order 2 extrapolation $\mathbf{a}^{n+1} \cdot \tau = -2\frac{\partial \phi^n}{\partial \tau} + \frac{\partial \phi^{n-1}}{\partial \tau}$ [7].

Algorithm 3.3 *(*Gauge Method (3.1.4) with Dirichlet Condition (3.1.8)*) Start with initial values* $\phi^0 = 0$ *and* $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$.

Step 1: *Find* $\mathbf{a}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n - \dfrac{1}{Re}\triangle \mathbf{a}^{n+1} = \mathbf{f}(t_{n+1}), & \text{in } \Omega, \\ \mathbf{a}^{n+1} \cdot \nu = -\dfrac{\partial \phi^n}{\partial \nu}, \quad \mathbf{a}^{n+1} \cdot \tau = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2.8}$$

Step 2: *Find* ϕ^{n+1} *as the solution of*

$$\begin{cases} -\triangle \phi^{n+1} = \operatorname{div} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \phi^{n+1} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2.9}$$

Step 3: *Find*

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla \phi^{n+1}, \quad \text{in } \Omega. \tag{3.2.10}$$

Algorithm 3.4 *(*Gauge Method (3.1.6) with Dirichlet Condition (3.1.8)*) Start with initial values* $\phi^0 = 0$ *and* $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$.

Step 1: *Find* $\mathbf{a}^{n+1}$ *as the solution of (3.2.8).*

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{cases} -\triangle \psi^{n+1} = \text{rot}\, \mathbf{a}^{n+1}, & \textit{in } \Omega, \\ \dfrac{\partial \psi^{n+1}}{\partial \nu} = 0, & \textit{on } \partial\Omega, \end{cases} \tag{3.2.11}$$

Step 3: *Find*

$$\begin{aligned} \mathbf{u}^{n+1} &= \mathbf{curl}\, \psi^{n+1}, && \textit{in } \Omega, \\ \nabla \phi^{n+1} &= \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, && \textit{in } \Omega. \end{aligned} \tag{3.2.12}$$

Remark 3.2 (Boundary Condition) In Algorithms 3.3 and 3.4, the boundary conditions of velocity $\mathbf{u}^{n+1}$ are

$$\mathbf{u}^{n+1} \cdot \nu = \frac{\partial \psi^{n+1}}{\partial \tau} = \frac{\partial \phi^{n+1}}{\partial \nu} - \frac{\partial \phi^{n}}{\partial \nu}, \quad \mathbf{u}^{n+1} \cdot \tau = 0, \quad \text{on } \partial\Omega. \tag{3.2.13}$$

On the other hand, after adding a suitable constant, $\phi^{n+1} = 0$ on $\partial\Omega$ in Algorithm 3.4. because of $\dfrac{\partial \psi^{n+1}}{\partial \nu} = 0$ on $\partial\Omega$. Consequently, ϕ^{n+1} of Algorithm 3.4 satisfies (3.2.9).

Remark 3.3 (Compatibility Condition) Upon integrating both sides of (3.2.9) and using (3.2.8), we uncover the relation

$$\begin{aligned} \int_{\Omega} \triangle \phi^{n+1} d\mathbf{x} &= \int_{\partial\Omega} \frac{\partial \phi^{n+1}}{\partial \nu} d\Gamma = -\int_{\partial\Omega} \mathbf{a}^{n+1} \cdot \nu d\Gamma \\ &= \int_{\partial\Omega} \frac{\partial \phi^{n}}{\partial \nu} d\Gamma = \int_{\Omega} \triangle \phi^{n} d\mathbf{x} \end{aligned} \tag{3.2.14}$$

for both Algorithms 3.3 and 3.4.

This means that $\int_\Omega \triangle \phi^n d\mathbf{x}$ is a constant for all time steps n, a property not generally valid. So, we cannot expect the numerical solution ϕ^n to converge to the exact solution ϕ. Since pressure p^n and ϕ^n are linked via (3.1.2), we cannot expect convergence of p^n to p. Therefore, Algorithms 3.3 and 3.4 cannot be used for calculating pressure p. However, the velocity $\mathbf{u}^{n+1}$ converges to exact solution $\mathbf{u}$ with a rate $\mathcal{O}(\sqrt{\triangle t})$; see Section 3.5.

All these Algorithms 3.1-3.4 have the stability constraint $C\triangle t \leq \frac{1}{Re}$ which limit their applicability to small to moderate Reynolds numbers. This will be proved in Section 3.3.1. These algorithms become unconditionally stable upon treating the convection term in the momentum equation semi-implicitly, namely,

$$\frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\triangle t} + (\mathbf{u}^n \cdot \nabla)\mathbf{a}^{n+1} + (\mathbf{u}^n \cdot \nabla)\nabla\phi^n - \frac{1}{Re}\triangle \mathbf{a}^{n+1} = \mathbf{f}(t_{n+1}). \qquad (3.2.15)$$

A proof is given in Section 3.3.2. We note that the formula (3.2.15) is still linear.

3.3 Stability

In this section, we examine the stability of Algorithms 3.1-3.4 and their semi-implicit variants. Algorithms 3.1-3.4 have the stability constraint $C\triangle t \leq \frac{1}{Re}$ which limit their applicability to small to moderate Reynolds numbers; this is proved in Subsection 3.3.1. In Subsection 3.3.2, we prove that these algorithms become unconditionally stable upon treating convection semi-implicitly in the momentum equation (3.2.15).

Since the time discrete function $\mathbf{u}^{n+1}$ does not vanish on the boundary, it cannot be used as a test function. So we introduce the auxiliary function $\widehat{\mathbf{u}}^{n+1}$ which is 0 on the boundary $\partial\Omega$:

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^n. \qquad (3.3.1)$$

In view of the boundary conditions (3.2.7) and (3.2.13) of $\mathbf{u}^{n+1}$ for the Neumann and Dirichlet conditions of ϕ^{n+1}, respectively, we can easily get the following useful properties:

Lemma 3.4 *Let n and m be non-negative integers. For both Neumann and Dirichlet boundary conditions, the following properties of $\widehat{\mathbf{u}}^n$ and $\mathbf{u}^n$ are valid:*

$$\widehat{\mathbf{u}}^n = 0, \quad on\ \partial\Omega, \tag{3.3.2}$$

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^n = \mathbf{u}^{n+1} - \nabla(\phi^{n+1} - \phi^n), \tag{3.3.3}$$

$$\langle \mathbf{u}^n\, ,\, \nabla\phi^m \rangle = 0, \quad and \quad \langle \widehat{\mathbf{u}}^n\, ,\, \mathbf{u}^m \rangle = \langle \mathbf{u}^n\, ,\, \mathbf{u}^m \rangle\,. \tag{3.3.4}$$

3.3.1 Explicit Convection Scheme

In order to estimate the convection term, we need to assume that the numerical solution is bounded in $\mathbf{L}^\infty(\Omega)$. This is a customary but rather strong assumption, which is removed in Section 3.3.2

Theorem 3.1 *Suppose*

$$\max_{0\le n\le N+1} \|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} < M. \tag{3.3.5}$$

If Assumptions 1-3 hold, and the stability constraint for the Algorithms 3.1-3.4

$$2M^2 Re \triangle t < 1 \tag{3.3.6}$$

is enforced, then the following a priori estimates are valid

$$\begin{aligned} \left\|\mathbf{u}^{N+1}\right\|_0^2 + \frac{\triangle t}{Re}\left\|\triangle\phi^{N+1}\right\|_0^2 \; &+ \; \frac{\triangle t}{4Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 \\ &\le \; \left\|\mathbf{u}^0\right\|_0^2 + C Re \triangle t \sum_{n=0}^{N} \|\mathbf{f}(t_{n+1})\|_{-1}^2. \end{aligned} \tag{3.3.7}$$

Proof. By definition of $\mathbf{u}^{n+1}$ and $\widehat{\mathbf{u}}^{n+1}$, the momentum equations for all explicit schemes can be rewritten as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} - \frac{\nabla(\phi^{n+1} - \phi^n)}{\triangle t} - \frac{1}{Re}\triangle(\widehat{\mathbf{u}}^{n+1} - \nabla\phi^n) = -(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n + \mathbf{f}(t_{n+1}). \tag{3.3.8}$$

We now multiply by $2\triangle t \widehat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and use Lemma 3.4 to get

$$\begin{aligned} &\left\|\mathbf{u}^{n+1}\right\|_0^2 - \|\mathbf{u}^n\|_0^2 + \left\|\mathbf{u}^{n+1} - \mathbf{u}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ +\frac{2\triangle t}{Re}\left\|\nabla \widehat{\mathbf{u}}^{n+1}\right\|_0^2 \;&=\; \frac{2\triangle t}{Re}\left\langle \triangle \phi^n \,,\, \mathrm{div}\, \widehat{\mathbf{u}}^{n+1}\right\rangle + 2\triangle t\left\langle \mathbf{f}(t_{n+1}) \,,\, \widehat{\mathbf{u}}^{n+1}\right\rangle \\ &\quad -2\triangle t \mathcal{N}(\mathbf{u}^n, \mathbf{u}^n, \widehat{\mathbf{u}}^{n+1}) = A_1 + A_2 + A_3. \end{aligned} \tag{3.3.9}$$

In view of Lemmas 1.9 and 3.4, we have $\|\triangle(\phi^{n+1} - \phi^n)\|_0^2 = \|\mathrm{div}\, \widehat{\mathbf{u}}^{n+1}\|_0^2 \leq \|\nabla \widehat{\mathbf{u}}^{n+1}\|_0^2$, whence

$$\begin{aligned} A_1 &= -\frac{2\triangle t}{Re}\left\langle \triangle \phi^n \,,\, \triangle(\phi^{n+1} - \phi^n)\right\rangle \\ &= -\frac{\triangle t}{Re}\left(\left\|\triangle \phi^{n+1}\right\|_0^2 - \|\triangle \phi^n\|_0^2 - \left\|\triangle(\phi^{n+1} - \phi^n)\right\|_0^2\right) \\ &\leq -\frac{\triangle t}{Re}\left(\left\|\triangle \phi^{n+1}\right\|_0^2 - \|\triangle \phi^n\|_0^2\right) + \frac{\triangle t}{Re}\left\|\nabla \widehat{\mathbf{u}}^{n+1}\right\|_0^2. \end{aligned} \tag{3.3.10}$$

Clearly,

$$\begin{aligned} A_2 &\leq C\triangle t \|\mathbf{f}(t_{n+1})\|_{-1}\left\|\widehat{\mathbf{u}}^{n+1}\right\|_1 \\ &\leq C Re \triangle t \|\mathbf{f}(t_{n+1})\|_{-1}^2 + \frac{\triangle t}{4Re}\left\|\nabla \widehat{\mathbf{u}}^{n+1}\right\|_0^2. \end{aligned} \tag{3.3.11}$$

Since $\langle (\mathbf{u}^n \cdot \nabla)\widehat{\mathbf{u}}^{n+1} \,,\, \mathbf{u}^n \rangle = \langle (\mathbf{u}^n \cdot \nabla)\widehat{\mathbf{u}}^{n+1} \,,\, \mathbf{u}^n - \widehat{\mathbf{u}}^{n+1} \rangle$ and in light of (3.3.3) and (3.3.4), $\mathbf{u}^n - \widehat{\mathbf{u}}^{n+1} = (\mathbf{u}^n - \mathbf{u}^{n+1}) + \nabla(\phi^{n+1} - \phi^n)$ is an orthogonal decomposition in $\mathbf{L}^2(\Omega)$, we have

$$\begin{aligned} A_3 &= 2\triangle t \left\langle (\mathbf{u}^n \cdot \nabla)\widehat{\mathbf{u}}^{n+1} \,,\, \mathbf{u}^n - \widehat{\mathbf{u}}^{n+1}\right\rangle \\ &\leq 2\triangle t \|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}\left\|\nabla \widehat{\mathbf{u}}^{n+1}\right\|_0\left\|\mathbf{u}^n - \widehat{\mathbf{u}}^{n+1}\right\|_0 \\ &\leq 2 Re M^2 \triangle t \left(\left\|\mathbf{u}^{n+1} - \mathbf{u}^n\right\|_0^2 + \left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2\right) \\ &\quad + \frac{\triangle t}{2Re}\left\|\nabla \widehat{\mathbf{u}}^{n+1}\right\|_0^2. \end{aligned} \tag{3.3.12}$$

Inserting (3.3.10)-(3.3.12) back into (3.3.9), we get

$$\begin{aligned}
&\left\|\mathbf{u}^{n+1}\right\|_0^2 - \|\mathbf{u}^n\|_0^2 + \frac{\triangle t}{Re}\left(\left\|\triangle\phi^{n+1}\right\|_0^2 - \|\triangle\phi^n\|_0^2\right) + \frac{\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 \\
&+(1-2ReM^2\triangle t)\left(\left\|\mathbf{u}^{n+1}-\mathbf{u}^n\right\|_0^2 + \left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2\right) \qquad (3.3.13) \\
&\leq CRe\triangle t\|\mathbf{f}(t_{n+1})\|_{-1}^2.
\end{aligned}$$

If the stability constraint (3.3.6) hold, then by adding over n from 0 to N, we obtain (3.3.7). ∎

3.3.2 Semi-Implicit Convection Scheme

By definition (3.3.1) of $\widehat{\mathbf{u}}^{n+1}$, the momentum equation (3.2.15) of Algorithms 3.1-3.4 can be rewritten as

$$\frac{\mathbf{a}^{n+1}-\mathbf{a}^n}{\triangle t} + (\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1} - \frac{1}{Re}\triangle\mathbf{a}^{n+1} = \mathbf{f}(t_{n+1}), \quad \text{in } \Omega. \qquad (3.3.14)$$

Upon multiplying (3.3.14) by $\widehat{\mathbf{u}}^{n+1}$, the convection term becomes $\mathcal{N}\left(\mathbf{u}^n, \widehat{\mathbf{u}}^{n+1}, \widehat{\mathbf{u}}^{n+1}\right) = 0$ because of Lemma 1.3. This is in striking contrast with Theorem 3.1 and the reason for removing assumption (3.3.5).

Theorem 3.2 *The scheme (3.3.14) is unconditionally stable for both Neumann and Dirichlet boundary conditions in the sense that for all $\triangle t > 0$ the following a priori bound holds:*

$$\begin{aligned}
&\sum_{n=0}^{N}\left(\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2 + \left\|\mathbf{u}^{n+1}-\mathbf{u}^n\right\|_0^2 + \frac{\triangle t}{Re}\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2\right) \\
&+\left\|\mathbf{u}^{N+1}\right\|_0^2 + \frac{\triangle t}{Re}\left\|\triangle\phi^{N+1}\right\|_0^2 \leq \left\|\mathbf{u}^0\right\|_0^2 + CRe\triangle t\sum_{n=0}^{N}\|\mathbf{f}(t_{n+1})\|_{-1}^2. \qquad (3.3.15)
\end{aligned}$$

PROOF. Since $\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}$ and $\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n}$, the momentum equation in (3.3.14) can be rewritten as

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\triangle t} - \frac{\nabla(\phi^{n+1} - \phi^n)}{\triangle t} &+ (\mathbf{u}^n \cdot \nabla)\widehat{\mathbf{u}}^{n+1} \\ &- \frac{1}{Re}\triangle(\widehat{\mathbf{u}}^{n+1} - \nabla\phi^n) = \mathbf{f}(t_{n+1}). \end{aligned} \tag{3.3.16}$$

We now take the inner product of (3.3.16) with $2\triangle t\widehat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$, and use Lemma 3.4 to get

$$\begin{aligned} &\left\|\mathbf{u}^{n+1}\right\|_0^2 - \left\|\mathbf{u}^n\right\|_0^2 + \left\|\mathbf{u}^{n+1} - \mathbf{u}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ &+\frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 = \frac{2\triangle t}{Re}\left\langle \triangle\phi^n \,, \operatorname{div} \widehat{\mathbf{u}}^{n+1}\right\rangle + 2\triangle t\left\langle \mathbf{f}(t_{n+1}) \,, \widehat{\mathbf{u}}^{n+1}\right\rangle . \end{aligned} \tag{3.3.17}$$

The convection vanishes by Lemma 1.3. On the other hand, proceeding as with (3.3.10) and (3.3.11) gives us

$$\begin{aligned} &\left\|\mathbf{u}^{n+1}\right\|_0^2 - \left\|\mathbf{u}^n\right\|_0^2 + \left\|\mathbf{u}^{n+1} - \mathbf{u}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ &+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 + \frac{\triangle t}{Re}\left(\left\|\triangle\phi^{n+1}\right\|_0^2 - \left\|\triangle\phi^n\right\|_0^2\right) \le C Re\triangle t\|\mathbf{f}(t_{n+1})\|_{-1}^2. \end{aligned}$$

On adding over n from 0 to N, we obtain (3.3.15). ■

3.4 A Priori Error Analysis for Velocity of Algorithms 3.1-3.2 with Neumann Condition

In this section, we carry out the error analysis for velocity of Algorithms 3.1-3.2, which employ Neumann condition. We first prove that the convergence rate of velocity is of order $\frac{1}{2}$, and then we improve the rate to order 1. We examine the semi-implicit momentum equation (3.2.15) in subsection 3.4.1 and the explicit convection in subsection 3.4.2. Since only the convection term is different between explicit and semi-implicit schemes, we just estimate the convection term in subsection 3.4.2.

We postpone the study of Algorithms 3.3-3.4 with Dirichlet condition until subsection 3.5, because of the orthogonality property $\langle \nabla q \,,\, \mathbf{u}^n \rangle = 0$.

Let $(\mathbf{u}(t_{n+1}), p(t_{n+1}))$ be the exact solution of (1.1.1) at the time step t_{n+1}. Let $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{a}^{n+1}, \phi^{n+1})$ be the numerical solution of any of the Algorithms 3.1-3.4. We will use the following notations:

$$\widehat{\mathbf{E}}^{n+1} = \mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}^{n+1}, \quad \mathbf{E}^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}, \quad e^{n+1} = p(t_{n+1}) - p^{n+1}.$$

We note again that $\widehat{\mathbf{u}}^{n+1} = 0$ on $\partial\Omega$ and div $\widehat{\mathbf{u}}^{n+1} \neq 0$ in Ω, whereas div $\mathbf{u}^{n+1} = 0$ in Ω and $\mathbf{u}^{n+1} = \nabla(\phi^{n+1} - \phi^n) \neq 0$ on $\partial\Omega$. The following lemma can be proved directly by invoking Lemma 3.4:

Lemma 3.5 (Properties of error functions) *Let n and m be non-negative integers. For both Neumann and Dirichlet boundary conditions, we have*

$$\text{div } \mathbf{E}^{n+1} = 0, \tag{3.4.1}$$

$$\widehat{\mathbf{E}}^{n+1} = 0, \quad \textit{on } \partial\Omega, \tag{3.4.2}$$

$$\widehat{\mathbf{E}}^{n+1} = \mathbf{E}^{n+1} + \nabla(\phi^{n+1} - \phi^n), \tag{3.4.3}$$

$$\langle \mathbf{E}^n \,,\, \nabla\phi^m \rangle = 0, \quad \textit{and} \quad \left\langle \widehat{\mathbf{E}}^n \,,\, \mathbf{E}^m \right\rangle = \langle \mathbf{E}^n \,,\, \mathbf{E}^m \rangle\,. \tag{3.4.4}$$

Using Lemmas 1.9 and 3.5, we have the following lemma:

Lemma 3.6 (Additional properties of Error Functions) *For both Neumann and Dirichlet boundary conditions, we have*

$$\left\| \triangle \left(\phi^{n+1} - \phi^n\right) \right\|_0^2 = \left\| \text{div } \widehat{\mathbf{E}}^{n+1} \right\|_0^2 \leq \left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0^2, \tag{3.4.5}$$

$$\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = \left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\nabla\left(\phi^{n+1}-\phi^n\right)\right\|_0^2, \tag{3.4.6}$$

and

$$\left\|\mathbf{E}^{n+1}\right\|_1^2 \le C\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\triangle\left(\phi^{n+1}-\phi^n\right)\right\|_0^2\right) \le C\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \tag{3.4.7}$$

We note that Lemmas 3.5 and 3.6 are satisfied also for Algorithms 3.3-3.4 with Dirichlet boundary condition.

3.4.1 Semi-Implicit Convection Scheme

In this subsection, we consider the semi-implicit momentum equation (3.2.15). We show that the numerical solution $\mathbf{u}^{n+1}$ of Algorithms 3.1-3.2 is an approximation of order $\frac{1}{2}$ to the exact solution $\mathbf{u}(t_{n+1})$ in $\mathbf{L}^2(\Omega)$ (Theorem 3.3). In Theorem 3.4, we improve the rate of convergence of Algorithms 3.1-3.2 to orders 1 weakly in $\mathbf{L}^2(\Omega)$ and strongly in $\mathbf{Z}^*$, respectively. The result of Theorem 3.3 is crucial to prove Theorem 3.4.

Theorem 3.3 *If Assumptions 1-3 hold, then the error functions of Algorithms 3.1-3.2 are bounded by*

$$\begin{aligned} \left\|\mathbf{E}^{N+1}\right\|_0^2 &+ \frac{\triangle t}{Re}\left\|\triangle\phi^{N+1}\right\|_0^2 + \frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &+ \sum_{n=0}^{N}\left(\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2 + \left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right) \le C\triangle t. \end{aligned} \tag{3.4.8}$$

Proof. By virtue of the Taylor theorem for the exact velocity $\mathbf{u}(t)$, we get

$$\begin{aligned} \frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\triangle t} &+ (\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1}) + \nabla p(t_{n+1}) \\ &- \frac{1}{Re}\triangle\mathbf{u}(t_{n+1}) = \mathbf{R}_{n+1} + \mathbf{f}(t_{n+1}), \end{aligned} \tag{3.4.9}$$

where $\mathbf{R}_{n+1} = \dfrac{1}{\triangle t}\displaystyle\int_{t_n}^{t_{n+1}} (t - t_n)\mathbf{u}_{tt}(t)dt$ is the truncation error. By subtracting (3.3.16) from (3.4.9)

$$\begin{aligned} &\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\triangle t} + \frac{\nabla(\phi^{n+1} - \phi^n)}{\triangle t} - \frac{1}{Re}\triangle\widehat{\mathbf{E}}^{n+1} - \frac{1}{Re}\nabla\triangle\phi^n \\ &\qquad = \mathbf{R}_{n+1} - \nabla p(t_{n+1}) - (\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1}) + (\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1}. \end{aligned} \tag{3.4.10}$$

Multiplying (3.4.10) by $2\triangle t\widehat{\mathbf{E}}^{n+1} \in \mathbf{H}_0^1(\Omega)$, and invoking Lemma 3.5, (3.4.10) becomes

$$\begin{aligned} &\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ &\qquad + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = 2\triangle t\left\langle \mathbf{R}_{n+1}\,,\,\widehat{\mathbf{E}}^{n+1}\right\rangle \\ &\qquad + 2\triangle t\left\langle p(t_{n+1})\,,\,\mathrm{div}\,\widehat{\mathbf{E}}^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle \triangle\phi^n\,,\,\mathrm{div}\,\widehat{\mathbf{E}}^{n+1}\right\rangle \\ &\qquad - 2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1}) - \mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{u}}^{n+1},\widehat{\mathbf{E}}^{n+1})\right) \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{3.4.11}$$

Since we get, by Hölder inequality,

$$\begin{aligned} \|\mathbf{R}_{n+1}\|_0^2 &\le C\frac{1}{\triangle t^2}\left\|\int_{t_n}^{t_{n+1}}(t - t_n)dt\int_{t_n}^{t_{n+1}}(t - t_n)\mathbf{u}_{tt}^2(t)dt\right\|_0^2 \\ &\le C\int_{t_n}^{t_{n+1}}\sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2dt, \end{aligned} \tag{3.4.12}$$

we deduce from (3.4.6)

$$\begin{aligned} A_1 &\le C\triangle t\|\mathbf{R}_{n+1}\|_0\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0 \\ &\le C\triangle t\int_{t_n}^{t_{n+1}}\sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2dt \\ &\quad + C\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2\right). \end{aligned} \tag{3.4.13}$$

By using Lemma 3.5 and the boundary values $\dfrac{\partial}{\partial \nu}(\phi^{n+1} - \phi^n) = 0$, we obtain

$$
\begin{aligned}
A_2 &= 2\triangle t \left\langle p(t_{n+1})\,,\ \triangle(\phi^{n+1} - \phi^n)\right\rangle \\
&= -2\triangle t \left\langle \nabla p(t_{n+1})\,,\ \nabla(\phi^{n+1} - \phi^n)\right\rangle \\
&\le C\triangle t^2 \|\nabla p(t_{n+1})\|_0^2 + \left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2.
\end{aligned}
\tag{3.4.14}
$$

We stress (3.4.14) is not valid for Algorithms 3.3-3.4 because of $\dfrac{\partial}{\partial \nu}(\phi^{n+1} - \phi^n) \neq 0$. Making use of Lemmas 3.5 and 3.6, we arrive at

$$
\begin{aligned}
A_3 &= -\frac{2\triangle t}{Re}\left\langle \triangle\phi^n\,,\ \triangle\left(\phi^{n+1} - \phi^n\right)\right\rangle \\
&= -\frac{\triangle t}{Re}\left(\left\|\triangle\phi^{n+1}\right\|_0^2 - \|\triangle\phi^n\|_0^2\right) + \frac{\triangle t}{Re}\left\|\triangle\left(\phi^{n+1} - \phi^n\right)\right\|_0^2 \\
&\le -\frac{\triangle t}{Re}\left(\left\|\triangle\phi^{n+1}\right\|_0^2 - \|\triangle\phi^n\|_0^2\right) + \frac{\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2.
\end{aligned}
\tag{3.4.15}
$$

We split the remaining term A_4, the only one dealing with convection, as follows:

$$
\begin{aligned}
A_4 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}^{n+1}) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{E}^n, \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}^{n+1}) - 2\triangle t\mathcal{N}(\mathbf{u}^n, \widehat{\mathbf{E}}^{n+1}, \widehat{\mathbf{E}}^{n+1}) \\
&= A_{4,1} + A_{4,2} + A_{4,3}.
\end{aligned}
\tag{3.4.16}
$$

Lemma 1.3 yields $A_{4,3} = 0$. The vanishing of $A_{4,3}$ is crucial to avoid boundedness of the numerical solution $\mathbf{u}^n$. We now resort to Lemmas 1.3 and 1.4 to estimate A_4. Since $\|\mathbf{u}(t_{n+1})\|_2 \le M$ by Lemma 1.5, we get

$$
\begin{aligned}
A_{4,1} &\le C\triangle t\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0 \\
&\le CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + \frac{\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2
\end{aligned}
\tag{3.4.17}
$$

as well as

$$\begin{aligned} A_{4,2} &\le C\triangle t\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0 \\ &\le CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.18}$$

Inserting (3.4.17)-(3.4.18) into (3.4.16) yields

$$A_4\le CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt+CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \tag{3.4.19}$$

Replacing (3.4.13)-(3.4.15), and (3.4.19) back into (3.4.11) implies

$$\begin{aligned} \left\|\mathbf{E}^{n+1}\right\|_0^2 &- \|\mathbf{E}^n\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2 \\ &+\frac{\triangle t}{Re}\left(\left\|\triangle\phi^{n+1}\right\|_0^2-\|\triangle\phi^n\|_0^2\right)+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &\le C\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2\right)+CRe\triangle t\|\mathbf{E}^n\|_0^2 \\ &+C\triangle t^2\|\nabla p(t_{n+1})\|_0^2+CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt \\ &+C\triangle t\int_{t_n}^{t_{n+1}}\sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2dt. \end{aligned} \tag{3.4.20}$$

Summing over n from 0 to N,

$$\begin{aligned} &\left\|\mathbf{E}^{N+1}\right\|_0^2+\frac{\triangle t}{Re}\left\|\triangle\phi^{N+1}\right\|_0^2+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &+\sum_{n=0}^{N}\left(\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right)\le CRe\triangle t\sum_{n=0}^{N}\|\mathbf{E}^n\|_0^2 \\ &+C\triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2\right)+C\triangle t^2\sum_{n=0}^{N}\|\nabla p(t_{n+1})\|_0^2 \\ &+CRe\triangle t^2\int_0^{t_{N+1}}\|\mathbf{u}_t(t)\|_0^2dt+C\triangle t\int_0^{t_{N+1}}\sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2dt. \end{aligned} \tag{3.4.21}$$

Finally, by the discrete Gronwall lemma and Lemma 1.5, we prove (3.4.8). ■

We observe that constant C in (3.4.8) depends exponentially on the Reynolds number Re. We also note that the suboptimal order $\frac{1}{2}$ of Theorem 3.3 is due to the pressure of $\|\nabla p(t_{n+1})\|_0$ in (3.4.14) and $C\triangle t \int_{t_n}^{t_{n+1}} \sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2 dt$ in (3.4.13). To improve the convergence rate we must get rid of these terms. This is precisely over next task. The main idea to obtain an error estimate in $L^2(\mathbf{L}^2)$ is to invert the main elliptic operator or, equivalently, multiply by a divergence free test function satisfying the Stokes equations. Let $(\mathbf{v}^{n+1}, q^{n+1}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be a solution of

$$\begin{cases} -\triangle \mathbf{v}^{n+1} + \nabla q^{n+1} = \mathbf{E}^{n+1}, & \text{in } \Omega, \\ \text{div } \mathbf{v}^{n+1} = 0, & \text{in } \Omega, \\ \mathbf{v}^{n+1} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.4.22}$$

In view of Assumption 1, $(\mathbf{v}^{n+1}, q^{n+1}) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ are strong solutions of (3.4.22) and satisfy

$$\left\|\mathbf{v}^{n+1}\right\|_2 + \left\|q^{n+1}\right\|_1 \leq C\left\|\mathbf{E}^{n+1}\right\|_0. \tag{3.4.23}$$

In particular, $\mathbf{v}^{n+1}$ satisfies the orthogonality $\langle \mathbf{v}^{n+1}, \nabla p\rangle = 0$, for all $\nabla p \in \mathbf{L}^2(\Omega)$. Since div $\mathbf{E}^{n+1} = 0$, which is stated in Lemma 3.5, we know $\langle \mathbf{E}^{n+1}, \nabla q^{n+1}\rangle = 0$, provided $\mathbf{E}^{n+1} \cdot \nu = 0$; this is the case of Algorithms 3.1 and 3.2 with Neumann condition. Now, we prove that the velocity error in Algorithms 3.1-3.2 is of order 1 weakly in $\mathbf{L}^2(\Omega)$ and strongly in $\mathbf{Z}^*$.

Theorem 3.4 *If Assumptions 1-3 hold, then the velocity error functions of Algorithms 3.1-3.2 satisfy*

$$\begin{aligned} \left\|\mathbf{E}^{N+1}\right\|_{\mathbf{Z}^*}^2 &+ \sum_{n=0}^{N} \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_{\mathbf{Z}^*}^2 \\ &+ \frac{\triangle t}{Re} \sum_{n=0}^{N} \left(\|\mathbf{E}_{n+1}\|_0^2 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \right) \leq M\triangle t^2. \end{aligned} \tag{3.4.24}$$

PROOF. Multiplying (3.4.10) by $2\triangle t\mathbf{v}^{n+1} \in \mathbf{H}_0^1(\Omega)$ gives us

$$2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n\,,\, \mathbf{v}^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{v}^{n+1}\right\rangle = 2\triangle t\left\langle \mathbf{R}_{n+1}\,,\,\mathbf{v}^{n+1}\right\rangle$$
$$-2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1}) + 2\triangle t\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{u}}^{n+1},\mathbf{v}^{n+1}). \tag{3.4.25}$$

Invoking (3.4.22), the leftest term can be treated as

$$\begin{aligned} 2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n\,,\, \mathbf{v}^{n+1}\right\rangle &= 2\left\langle \nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\,,\,\nabla\mathbf{v}^{n+1}\right\rangle \\ &= \left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 - \left\|\nabla\mathbf{v}^n\right\|_0^2 + \left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\|_0^2, \end{aligned} \tag{3.4.26}$$

whereas the next term can be written as, with the aid of Lemma 3.5,

$$\begin{aligned} &\frac{2\triangle t}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{v}^{n+1}\right\rangle = \frac{2\triangle t}{Re}\left\langle \widehat{\mathbf{E}}^{n+1}\,,\,\mathbf{E}^{n+1} - \nabla q^{n+1}\right\rangle \\ &= \frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 + \frac{2\triangle t}{Re}\left\langle \nabla(\phi^{n+1}-\phi^n)\,,\,\nabla q^{n+1}\right\rangle. \end{aligned} \tag{3.4.27}$$

Here we have the orthogonality $\langle \mathbf{E}^{n+1}\,,\,\nabla q\rangle = 0$, which is valid for Algorithms 3.1 and 3.2 because $\mathbf{E}^{n+1}\cdot\nu = \dfrac{\partial}{\partial\nu}(\phi^{n+1}-\phi^n) = 0$. Collecting the above results, (3.4.25) becomes

$$\begin{aligned} &\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 - \left\|\nabla\mathbf{v}^n\right\|_0^2 + \left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\ &= 2\triangle t\left\langle \mathbf{R}_{n+1}\,,\,\mathbf{v}^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle \widehat{\mathbf{E}}^{n+1}\,,\,\nabla q^{n+1}\right\rangle \\ &\quad +2\triangle t\left(\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{u}}^{n+1},\mathbf{v}^{n+1}) - \mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1})\right) \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{3.4.28}$$

Since $\mathbf{v}^{n+1} \in \mathbf{Z}(\Omega)$, we readily have

$$A_1 \;\leq\; C\triangle t^2\int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|_{\mathbf{Z}^*}^2\,dt + \mathbf{C}\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2. \tag{3.4.29}$$

In view of (3.4.23), we get

$$\begin{aligned} A_2 &= \frac{2\triangle t}{Re}\left\langle \nabla(\phi^{n+1}-\phi^n)\,,\,\nabla q^{n+1}\right\rangle \\ &\le \frac{C\triangle t}{Re}\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2+\frac{\triangle t}{2Re}\left\|\mathbf{E}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.30}$$

The convection term A_3 can be rewritten as follows:

$$\begin{aligned} A_3 &= -2\triangle t\mathcal{N}(\mathbf{E}^n,\widehat{\mathbf{u}}^{n+1},\mathbf{v}^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\widehat{\mathbf{u}}^{n+1},\mathbf{v}^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1},\mathbf{v}^{n+1}) = A_{3,1}+A_{3,2}+A_{3,3}. \end{aligned} \tag{3.4.31}$$

We now proved to estimate each term in (3.4.31) respectively. We first note that

$$\begin{aligned} A_{3,1} &= 2\triangle t\mathcal{N}(\mathbf{E}^n,\widehat{\mathbf{E}}^{n+1},\mathbf{v}^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1}) = A_{3,1,1}+A_{3,1,2}. \end{aligned} \tag{3.4.32}$$

Since Theorem 3.3 yields $\|\mathbf{E}^n\|_0 \le C\triangle t^{\frac{1}{2}}$, using (3.4.23) we get

$$\begin{aligned} A_{3,1,1} &\le C\triangle t\|\mathbf{E}^n\|_0\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{v}^{n+1}\right\|_2 \\ &\le C\triangle t^{\frac{3}{2}}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{E}^{n+1}\right\|_0 \\ &\le CRe\triangle t^2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\frac{\triangle t}{8Re}\left\|\mathbf{E}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.33}$$

Since Lemma 1.5 gives $\|\mathbf{u}(t_{n+1})\|_2 \le C$, we easily deduce

$$\begin{aligned} A_{3,1,2} &\le C\triangle t\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}^{n+1}\right\|_0 \\ &\le \frac{\triangle t}{8Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right)+CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.34}$$

On other hand, we shift $A_{3,2}$ as follows

$$\begin{aligned} A_{3,2} &= 2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1},\mathbf{v}^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1}) = A_{3,2,1}+A_{3,2,2}, \end{aligned} \tag{3.4.35}$$

and make use again of $\|\mathbf{u}(t_{n+1})\|_2 \leq C$ to obtain

$$\begin{aligned} A_{3,2,1} &\leq C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\nabla\mathbf{v}^{n+1}\right\|_0 \\ &\leq \frac{\triangle t}{8Re}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.36}$$

Likewise,

$$\begin{aligned} A_{3,2,2} &\leq C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}^{n+1}\right\|_0 \\ &\leq C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + C\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.37}$$

Since div $\mathbf{u}(t_{n+1})=0$, we exchange the last two arguments of $A_{3,3}$ to write

$$\begin{aligned} A_{3,3} &\leq C\triangle t\|\mathbf{u}(t_{n+1})\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\nabla\mathbf{v}^{n+1}\right\|_0 \\ &\leq \frac{\triangle t}{8Re}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2. \end{aligned} \tag{3.4.38}$$

Inserting (3.4.32)-(3.4.38) into (3.4.31) yields

$$\begin{aligned} A_3 \leq\ & \frac{\triangle t}{2Re}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\triangle t}{8Re}\|\mathbf{E}^{n+1}-\mathbf{E}^n\|_0^2 + CRe\triangle t^2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &+CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 + C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt. \end{aligned} \tag{3.4.39}$$

Since $\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = \|\mathbf{E}^{n+1}\|_0^2 + \|\nabla(\phi^{n+1}-\phi^n)\|_0^2$, combining (3.4.29)-(3.4.30) with (3.4.39) leads to

$$\begin{aligned} &\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 - \|\nabla\mathbf{v}^n\|_0^2 + \left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\|_0^2 + \frac{\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\ &\leq CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 + \frac{C\triangle t}{Re}\left(\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2 + \left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right) \\ &\quad +CRe\triangle t^2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + C\triangle t^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*}^2 + \|\mathbf{u}_t(t)\|_0^2\right)dt. \end{aligned} \tag{3.4.40}$$

On adding over n from 0 to N, and recalling that $\mathbf{v}^0 = 0$,

$$\begin{aligned}
&\left\|\nabla \mathbf{v}^{N+1}\right\|_0^2 + \sum_{n=0}^{N} \left\|\nabla\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
\leq \quad & CRe\triangle t \sum_{n=0}^{N}\left\|\nabla \mathbf{v}^{n+1}\right\|_0^2 + CRe\triangle t^2 \sum_{n=0}^{N}\left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\
& + \frac{C\triangle t}{Re}\sum_{n=0}^{N}\left(\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2\right) \\
& + C\triangle t^2 \int_0^{t_{N+1}} \left(\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*}^2 + \|\mathbf{u}_t(t)\|_0^2\right) dt.
\end{aligned} \tag{3.4.41}$$

Combining the weak estimate

$$\begin{aligned}
&\frac{1}{Re}\sum_{n=0}^{N}\left(\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2\right) \\
&+\triangle t \sum_{n=0}^{N}\left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2 \leq M\triangle t.
\end{aligned}$$

From Theorem 3.3 with Lemmas 1.5 and 1.7, we have

$$\begin{aligned}
&\left\|\nabla \mathbf{v}^{N+1}\right\|_0^2 + \sum_{n=0}^{N} \left\|\nabla\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
&\leq CRe\triangle t \sum_{n=0}^{N}\left\|\nabla \mathbf{v}^{n+1}\right\|_0^2 + M\triangle t^2.
\end{aligned} \tag{3.4.42}$$

Applying the discrete Gronwall lemma, we reduce (3.4.42) to

$$\left\|\nabla \mathbf{v}^{N+1}\right\|_0^2 + \sum_{n=0}^{N} \left\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2 \leq M\triangle t^2. \tag{3.4.43}$$

Also, using Theorem 3.3, we get

$$\triangle t \sum_{n=0}^{N}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \leq \triangle t \sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\nabla\left(\phi^{n+1} - \phi^n\right)\right\|_0^2\right) \leq M\triangle t^2. \tag{3.4.44}$$

Therefore, we finally get (3.4.24) by using Lemma 1.1. ∎

3.4.2 Explicit Convection Scheme

In this section, we carry out the error analysis for the explicit convection scheme. Since the argument is similar to that of Section 3.4.1, we force on the explicit convection term. To this end, we need the additional assumption that the numerical solution bounded in $\mathbf{L}^\infty(\Omega)$. This assumption is also made in [30]

We first prove that Algorithms 3.1-3.2 are of order $\frac{1}{2}$ strongly, which mimics the results of Theorem 3.3.

Theorem 3.5 *Suppose*

$$\max_{0\le n\le N+1} \|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} < M, \tag{3.4.45}$$

and Assumptions 1-3 hold. Then Algorithms 3.1-3.2 satisfy (3.4.8).

PROOF. Instead of (3.4.16) in the proof of Theorem 3.3, the convection term A_4 is split as follows:

$$\begin{aligned}
\widehat{A}_4 &= -2\triangle t\Big(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1}) - \mathcal{N}(\mathbf{u}^n,\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1})\Big)\\
&= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}^n,\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}^n,\mathbf{E}^n,\widehat{\mathbf{E}}^{n+1}) = \widehat{A}_{4,1}+\widehat{A}_{4,2}+\widehat{A}_{4,3}+\widehat{A}_{4,4}.
\end{aligned} \tag{3.4.46}$$

By Lemma 1.4 and $\|\mathbf{u}(t_{n+1})\|_2 \le M$, we obtain

$$\begin{aligned}
\widehat{A}_{4,1} &\le C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\\
&\le CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt + \frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2,
\end{aligned} \tag{3.4.47}$$

as well as

$$\begin{aligned}\widehat{A}_{4,2} &\leq C\triangle t\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0 \\ &\leq CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2.\end{aligned} \tag{3.4.48}$$

Since $\operatorname{div}\mathbf{u}^n = 0$, Lemma 1.3 yield $\widehat{A}_{4,3} = 2\triangle t\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1},\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))$. Invoking $\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} < M$, we get

$$\begin{aligned}\widehat{A}_{4,3} &\leq C\triangle t\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0 \\ &\leq \frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2.\end{aligned} \tag{3.4.49}$$

Likewise $\widehat{A}_{4,4} = 2\triangle t\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1},\mathbf{E}^n)$ and

$$\begin{aligned}\widehat{A}_{4,4} &\leq C\triangle t\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{E}^n\|_0\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0 \\ &\leq CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2.\end{aligned} \tag{3.4.50}$$

So,

$$\widehat{A}_4 \leq CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt+CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \tag{3.4.51}$$

Since this estimate for $\widehat{A}_4$ is the same as for A_4 in (3.4.19), we proved as in Theorem 3.3 to conclude (3.4.8). ■

We finally establish that Algorithms 3.1-3.2 are order 1 weakly, there by extending Theorem 3.4 for explicit convection schemes.

Theorem 3.6 *Suppose*

$$\max_{0\leq n\leq N+1}\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} < M, \tag{3.4.52}$$

and Assumptions 1-3 hold. Then Algorithms 3.1-3.2 satisfy (3.4.24).

PROOF. Instead of (3.4.31) in the proof of Theorem 3.4, the convection term A_3 becomes

$$\begin{aligned}
\widehat{A}_3 &= -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1}) - \mathcal{N}(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}^{n+1})\right)\\
&= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}^n,\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{v}^{n+1})\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}^n,\mathbf{E}^n,\mathbf{v}^{n+1})\\
&= \widehat{A}_{3,1}+\widehat{A}_{3,2}+\widehat{A}_{3,3}+\widehat{A}_{3,4}.
\end{aligned} \tag{3.4.53}$$

We now argue as in Theorem 3.5. We observe first Lemmas 1.4-1.5 yield

$$\begin{aligned}
\widehat{A}_{3,1} &\le C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}^{n+1}\right\|_0\\
&\le C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + C\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2,
\end{aligned} \tag{3.4.54}$$

and

$$\begin{aligned}
\widehat{A}_{3,2} &\le \triangle t\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}^{n+1}\right\|_0\\
&\le \frac{\triangle t}{16Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right)+CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2.
\end{aligned} \tag{3.4.55}$$

Secondly, we use Lemmas 1.3 and 1.5 to infer that

$$\begin{aligned}
\widehat{A}_{3,3} &\le C\triangle t\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\left\|\nabla\mathbf{v}^{n+1}\right\|_0\\
&\le C\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2 + C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt,
\end{aligned} \tag{3.4.56}$$

and

$$\begin{aligned}
\widehat{A}_{3,4} &\le C\triangle t\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{E}^n\|_0\left\|\nabla\mathbf{v}^{n+1}\right\|_0\\
&\le \frac{\triangle t}{16Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right)+CRe\triangle t\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2.
\end{aligned} \tag{3.4.57}$$

So, (3.4.53) becomes

$$\begin{aligned}\widehat{A}_3 \quad &\leq \quad C\triangle t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_0^2 dt + CRe\triangle t \left\|\nabla \mathbf{v}^{n+1}\right\|_0^2 \\ &\quad + \frac{\triangle t}{8Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^2\right\|_0^2\right).\end{aligned} \tag{3.4.58}$$

Making use of (3.4.58) in place of (3.4.39), and proceeding as in Theorem 3.5, we easily arrive at (3.4.24). ∎

3.5 A Priori Error Analysis for Velocity : Algorithms 3.3-3.4 with Dirichlet Condition

In this section, we estimate the velocity error for Algorithms 3.3-3.4, which employ a Dirichlet condition. The chief difficulty of this estimate is that now $\mathbf{u}^n \cdot \nu \neq 0$. This is responsible for the reduced order $\mathcal{O}(\sqrt{\triangle t})$ of Theorem 3.7, which is consistent with a rate obtained in [30] via asymptotic, and for the request of the additional regularity assumption (3.5.1). However, (3.5.1) is a weaker condition than (2.2.7) in Chorin-Uzawa method.

Theorem 3.7 *Let Assumptions 1-3 hold, If, for* $0 \leq T \leq \infty$*,*

$$\int_0^T \|p_t(t)\|_0^2 dt \leq M, \tag{3.5.1}$$

then the error functions of Algorithms 3.3-3.4 with semi-implicit convection (3.2.15) satisfy

$$\begin{aligned}&\left\|\mathbf{E}^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2\right) \\ &\quad + Re\triangle t\left\|p(t_{N+1}) - \frac{1}{Re}\triangle\phi^{N+1}\right\|_0^2 + \frac{\triangle t}{4Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \leq C\triangle t.\end{aligned} \tag{3.5.2}$$

PROOF. The departing point is again (3.4.11) which we rewrite here for convenience:

$$\begin{aligned}\left\|\mathbf{E}^{n+1}\right\|_0^2 & -\|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ & + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = 2\triangle t\left\langle \mathbf{R}_{n+1}\,,\,\widehat{\mathbf{E}}^{n+1}\right\rangle \\ & + 2\triangle t\left\langle p(t_{n+1})\,,\,\mathrm{div}\,\widehat{\mathbf{E}}^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle \triangle\phi^n\,,\,\mathrm{div}\,\widehat{\mathbf{E}}^{n+1}\right\rangle \\ & - 2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}^{n+1}) - \mathcal{N}(\mathbf{u}^n, \widehat{\mathbf{u}}^{n+1}, \widehat{\mathbf{E}}^{n+1})\right) \\ = & \; A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{3.5.3}$$

To estimate A_1 and A_4 we proved as in Theorem 3.3 and thereby obtain (3.4.13) and (3.4.19), respectively. The remaining two terms A_2 and A_3 are more delicate and are handled together as follows:

$$\begin{aligned} A_2 + A_3 & = 2\triangle t\left\langle p(t_{n+1}) - \frac{1}{Re}\triangle\phi^n\,,\,\triangle(\phi^{n+1} - \phi^n)\right\rangle \\ & = 2\triangle t\left\langle p(t_{n+1}) - p(t_n)\,,\,\triangle(\phi^{n+1} - \phi^n)\right\rangle \\ & \quad -2Re\triangle t\left\langle q^n\,,\,q^{n+1} - q^n\right\rangle + 2Re\triangle t\left\langle q^n\,,\,p(t_{n+1}) - p(t_n)\right\rangle \\ & = B_1 + B_2 + B_3, \end{aligned} \tag{3.5.4}$$

where $q^n = p(t_n) - \frac{1}{Re}\triangle\phi^n$. In view of (3.5.1) we have $\|p(t_{n+1}) - p(t_n)\| \leq \left(\triangle t\int_{t_n}^{t_{n+1}} \|p_t(t)\|_0^2 dt\right)^{\frac{1}{2}}$, whence using Lemma 3.6,

$$\begin{aligned} B_1 & \leq C\triangle t\|p(t_{n+1}) - p(t_n)\|_0\left\|\triangle(\phi^{n+1} - \phi^n)\right\|_0 \\ & \leq CRe\triangle t^2\int_{t_n}^{t_{n+1}} \|p_t(t)\|_0^2 dt + \frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \end{aligned} \tag{3.5.5}$$

and

$$B_3 \leq CRe\triangle t^2\|q^n\|_0^2 + CRe\triangle t\int_{t_n}^{t_{n+1}} \|p_t(t)\|_0^2 dt. \tag{3.5.6}$$

For B_2 we employ, in addition, the elementary inequality $(a+b)^2 \le (1+\varepsilon)a^2 + (1+\frac{1}{\varepsilon})b^2$, for $\varepsilon > 0$,

$$\begin{aligned} B_2 &= -Re\triangle t\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2\right) \\ &\quad +Re\triangle t\left\|(p(t_{n+1}) - p(t_n)) - \frac{1}{Re}\triangle(\phi^{n+1} - \phi^n)\right\|_0^2 \\ &\le -Re\triangle t\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2\right) \\ &\quad +CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2 dt + \frac{9\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \end{aligned} \tag{3.5.7}$$

From (3.5.5)-(3.5.6), we get

$$\begin{aligned} A_2 + A_3 \le &\ -Re\triangle t\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2\right) + CRe\triangle t^2\|q^n\|_0^2 \\ &+\frac{5\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + CRe\triangle t\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2 dt. \end{aligned} \tag{3.5.8}$$

Inserting (3.4.13), (3.4.19), and (3.5.8) into (3.5.3) yields

$$\begin{aligned} &\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + 2\left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2 \\ &+Re\triangle t\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2\right) + \frac{\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &\le C\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\nabla(\phi^{n+1} - \phi^n)\right\|_0^2\right) + CRe\triangle t\|\mathbf{E}^n\|_0^2 \\ &\quad +CRe\triangle t^2\|q^n\|_0^2 + CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt \\ &\quad +CRe\triangle t\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2 dt + C\triangle t\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2 dt. \end{aligned} \tag{3.5.9}$$

Summing over n from 0 to N,

$$\left\|\mathbf{E}^{N+1}\right\|_0^2+\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2\right)$$

$$+Re\triangle t\left\|q^{N+1}\right\|_0^2+\frac{\triangle t}{4Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2$$

$$\leq C\triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla(\phi^{n+1}-\phi^n)\right\|_0^2\right)+CRe\triangle t\sum_{n=0}^{N}\|\mathbf{E}^n\|_0^2 \tag{3.5.10}$$

$$+Re\triangle t\|p(t_0)\|_0^2+CRe\triangle t^2\sum_{n=0}^{N}\|q^n\|_0^2+C\triangle t\int_0^{t_{N+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt$$

$$+CRe\triangle t^2\int_0^{t_{N+1}}\|\mathbf{u}_t(t)\|_0^2dt+CRe\triangle t\int_0^{t_{N+1}}\|p_t(t)\|_0^2dt.$$

Finally, by the discrete Gronwall lemma, we conclude (3.5.2). ■

Arguing as in Theorems 3.5 and 3.7, we can get the following result for explicit convection schemes.

Theorem 3.8 *Let Assumptions 1-3 hold, If, for* $0\leq T\leq\infty$

$$\max_{0\leq n\leq N+1}\|\mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)}<M \quad \textit{and} \quad \int_0^T\|p_t(t)\|_0^2dt\leq M, \tag{3.5.11}$$

then the error functions of Algorithms 3.3-3.4 (with explicit convection) satisfy (3.5.2).

3.6 A Priori Error Analysis for Pressure

The goal of this section is to estimate the pressure error in $L^2(L^2)$ for Algorithms 3.1 and 3.2. Recall from (3.2.6) that the discrete pressure p^{n+1} is given by

$$p^{n+1}=-\frac{\phi^{n+1}-\phi^n}{\triangle t}+\frac{1}{Re}\triangle\phi^{n+1}. \tag{3.6.1}$$

The main auxiliary bound to derive a pressure error estimate is the following:

$$\triangle t \sum_{n=0}^{N} \left\| \mathbf{E}^{n+1} - \mathbf{E}^{n} \right\|_0^2 \le C \triangle t^3. \tag{3.6.2}$$

To achieve this, we need additional regularity assumptions for the exact solution of (1.1.1) beyond Assumptions 1-3 in section 3.4.

Lemma 3.7 *Suppose Assumptions 1-3 hold. If*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le C, \tag{3.6.3}$$

then we have

$$\left\|\mathbf{E}^1\right\|_0^2 + \left\|\nabla \phi^1\right\|_0^2 + \frac{\triangle t}{Re}\left\|\triangle \phi^1\right\|_0^2 + \frac{\triangle t}{2Re}\left\|\nabla \widehat{\mathbf{E}}^1\right\|_0^2 \le C \triangle t^2 \tag{3.6.4}$$

and

$$\left\|\nabla \mathbf{v}^1\right\|_0^2 \le C \triangle t^3. \tag{3.6.5}$$

PROOF. By choosing $n = 0$ in (3.4.20), and realizing that $\mathbf{E}^0 = 0$ and $\phi^0 = 0$, we have

$$\begin{aligned} &\left\|\mathbf{E}^1\right\|_0^2 + \left\|\nabla \phi^1\right\|_0^2 + \frac{\triangle t}{Re}\left\|\triangle \phi^1\right\|_0^2 + \frac{\triangle t}{2Re}\left\|\nabla \widehat{\mathbf{E}}^1\right\|_0^2 \\ &\le C\triangle t \left(\left\|\mathbf{E}^1\right\|_0^2 + \left\|\nabla \phi^1\right\|_0^2\right) + C\triangle t^2 \|\nabla p(t_1)\|_0^2 \\ &\quad + CRe\triangle t^2 \int_0^{t_1} \|\mathbf{u}_t(t)\|_0^2 dt + C\triangle t \int_0^{t_1} \sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2 dt. \end{aligned} \tag{3.6.6}$$

Since $\sigma(t) = t$ in $t \in [0,1]$, we get (3.6.4) from (3.6.3) and Lemma 1.6. Since $\|\nabla \mathbf{v}^1\|_0 \le \|\mathbf{E}^1\|_0$, we can get, by choosing $n = 0$ in (3.4.40),

$$\begin{aligned} \left\|\nabla \mathbf{v}^1\right\|_0^2 &\le CRe\triangle t\left\|\nabla \mathbf{v}^1\right\|_0^2 + \frac{C\triangle t}{Re}\left(\left\|\nabla \phi^1\right\|_0^2 + \left\|\mathbf{E}^1\right\|_0^2\right) \\ &\quad + CRe\triangle t^2 \left\|\nabla \widehat{\mathbf{E}}^1\right\|_0^2 + C\triangle t^2 \int_0^{t_1} \left(\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*}^2 + \|\mathbf{u}_t(t)\|_0^2\right) dt. \end{aligned} \tag{3.6.7}$$

We note $\|\mathbf{u}_{tt}\|_{\mathbf{Z}^*}^2 + \|\mathbf{u}_t(t)\|_0^2 \le C$ by Lemmas 1.5 and 1.7. Finally (3.6.3) and Lemma 1.6 imply (3.6.5). ■

Now, we prove $\left\|\mathbf{E}^{N+1} - \mathbf{E}^N\right\|_0^2 \le C\triangle t^2$. This will be used in proof of the main result (3.6.2) in Lemma 3.9.

Lemma 3.8 *Suppose Assumptions 1-3 hold. If*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le C, \tag{3.6.8}$$

then the error functions of Algorithms 3.1-3.2 are bounded by

$$\begin{aligned} &\left\|\mathbf{E}^{N+1} - \mathbf{E}^N\right\|_0^2 + \frac{\triangle t}{Re}\left\|\triangle(\phi^{N+1} - \phi^N)\right\|_0^2 \\ &+\frac{\triangle t}{4Re}\sum_{n=1}^{N}\left\|\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\right\|_0^2 + \sum_{n=1}^{N}\left\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right\|_0^2 \\ &+2\sum_{n=1}^{N}\left\|\nabla(\phi^{n+1} - 2\phi^n + \phi^{n-1})\right\|_0^2 \le C\triangle t^2. \end{aligned} \tag{3.6.9}$$

Proof. Subtracting the n-th formula (3.4.10) from $(n+1)$-th one, we have

$$\begin{aligned} &\frac{\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}}{\triangle t} + \frac{\nabla(\phi^{n+1} - 2\phi^n - \phi^{n-1})}{\triangle t} - \frac{1}{Re}\triangle(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n) \\ &= (\mathbf{R}_{n+1} - \mathbf{R}_n) - \nabla(p(t_{n+1}) - p(t_n)) + \frac{1}{Re}\nabla\triangle(\phi^n - \phi^{n-1}) \\ &\quad -(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1}) + (\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1} \\ &\quad +(\mathbf{u}(t_n)\cdot\nabla)\mathbf{u}(t_n) - (\mathbf{u}^{n-1}\cdot\nabla)\widehat{\mathbf{u}}^n. \end{aligned} \tag{3.6.10}$$

We multiply by $2\triangle t(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}) = 2\triangle t\left(\mathbf{E}^{n+1}-\mathbf{E}^{n}+\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right) \in \mathbf{H}_0^1(\Omega)$, and Lemma 3.5 to get

$$
\begin{aligned}
&\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2-\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2\\
&+2\left\|\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right\|_0^2+\frac{2\triangle t}{Re}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right)\right\|_0^2\\
&=2\triangle t\left\langle \mathbf{R}_{n+1}-\mathbf{R}_{n}\,,\,\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right\rangle\\
&\quad+2\triangle t\left\langle p(t_{n+1})-p(t_n)\,,\,\operatorname{div}\,(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\rangle\\
&\quad-\frac{2\triangle t}{Re}\left\langle \triangle(\phi^{n}-\phi^{n-1})\,,\,\operatorname{div}\,(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\rangle\\
&\quad-2\triangle t\Big(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\\
&\qquad\qquad\qquad-\mathcal{N}(\mathbf{u}^{n},\widehat{\mathbf{u}}^{n+1},\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\Big)\\
&\quad+2\triangle t\Big(\mathcal{N}(\mathbf{u}(t_n),\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})-\mathcal{N}(\mathbf{u}^{n-1},\widehat{\mathbf{u}}^{n},\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\Big)\\
&=A_1+A_2+A_3+A_4+A_5.
\end{aligned}
\tag{3.6.11}
$$

We now estimate each term A_i separately. First, we observe that

$$
A_1 \;\leq\; \frac{\triangle t}{8Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2+CRe\triangle t^2\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt, \tag{3.6.12}
$$

and

$$
A_2 \;\leq\; \frac{\triangle t}{8Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2+CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2dt. \tag{3.6.13}
$$

We next use Lemmas 3.5 and 3.6 to obtain

$$
\begin{aligned}
A_3 &= -\frac{2\triangle t}{Re}\left\langle \triangle(\phi^n-\phi^{n-1}),\, \triangle(\phi^{n+1}-2\phi^n+\phi^{n-1})\right\rangle \\
&= -\frac{\triangle t}{Re}\left(\left\|\triangle(\phi^{n+1}-\phi^n)\right\|_0^2-\left\|\triangle(\phi^n-\phi^{n-1})\right\|_0^2\right) \\
&\quad +\frac{\triangle t}{Re}\left\|\triangle(\phi^{n+1}-2\phi^n+\phi^{n-1})\right\|_0^2 \\
&\leq -\frac{\triangle t}{Re}\left(\left\|\triangle(\phi^{n+1}-\phi^n)\right\|_0^2-\left\|\triangle(\phi^n-\phi^{n-1})\right\|_0^2\right) \\
&\quad +\frac{\triangle t}{Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2.
\end{aligned}
\tag{3.6.14}
$$

We can split the convection terms as follows: By $\|\mathbf{u}(t_n)\|_2 \leq C$ and Lemma 1.4,

$$
\begin{aligned}
A_4 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1},\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&= A_{4,1}+A_{4,2}+A_{4,3},
\end{aligned}
\tag{3.6.15}
$$

and

$$
\begin{aligned}
A_5 &= 2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{E}^{n-1},\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{u}^{n-1},\widehat{\mathbf{E}}^n,\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n) \\
&= A_{5,1}+A_{5,2}+A_{5,3}.
\end{aligned}
\tag{3.6.16}
$$

Since $\|\mathbf{u}(t_n)\|_2 \leq M$ from Lemma 1.5, Lemma 1.4 yields

$$
\begin{aligned}
A_{4,1} &\leq C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_1 \\
&\leq CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt+\frac{\triangle t}{20Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2,
\end{aligned}
\tag{3.6.17}
$$

$$\begin{aligned} A_{4,2} &\le C\triangle t\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_1 \\ &\le CRe\triangle t\|\mathbf{E}^n\|_0^2+\frac{\triangle t}{20Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0, \end{aligned} \tag{3.6.18}$$

$$\begin{aligned} A_{5,1} &\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n-1})\|_0\|\mathbf{u}(t_n)\|_2\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right)\right\|_0 \\ &\le CRe\triangle t^2\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_0^2+\frac{\triangle t}{20Re}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right)\right\|_0^2, \end{aligned} \tag{3.6.19}$$

$$\begin{aligned} A_{5,2} &\le \triangle t\left\|\mathbf{E}^{n-1}\right\|_0\|\mathbf{u}(t_n)\|_2\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0 \\ &\le CRe\triangle t\left\|\mathbf{E}^{n-1}\right\|_0^2+\frac{\triangle t}{20Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2. \end{aligned} \tag{3.6.20}$$

Invoking the crucial properties of $\mathcal{N}$ of Lemma 1.3, we infer that

$$\begin{aligned} A_{4,3}+A_{5,3} &= 2\triangle t\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1},\widehat{\mathbf{E}}^n)+2\triangle t\mathcal{N}(\mathbf{u}^{n-1},\widehat{\mathbf{E}}^n,\widehat{\mathbf{E}}^{n+1}) \\ &= 2\triangle t\mathcal{N}(\mathbf{u}^n-\mathbf{u}^{n-1},\widehat{\mathbf{E}}^{n+1},\widehat{\mathbf{E}}^n) \\ &= 2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n,\widehat{\mathbf{E}}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{E}^n-\mathbf{E}^{n-1},\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n,\widehat{\mathbf{E}}^n) \\ &= B_1^n+B_2^n. \end{aligned} \tag{3.6.21}$$

We postpone the estimate of B_2^n until the end of the proof. Since $\left\|\widehat{\mathbf{E}}^n\right\|_1\le C$,

$$\begin{aligned} B_1^n &\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n-1})\|_1\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 \\ &\le CRe\triangle t^2\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_1^2dt+\frac{\triangle t}{20Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0. \end{aligned} \tag{3.6.22}$$

So, collect (3.6.15)-(3.6.21), we have

$$\begin{aligned} A_4+A_5 &\le B_2^n+CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\left\|\mathbf{E}^{n-1}\right\|_0^2\right) \\ &\quad +\frac{\triangle t}{4Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+CRe\triangle t^2\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_t(t)\|_1^2dt. \end{aligned} \tag{3.6.23}$$

Inserting (3.6.12)-(3.6.14) and (3.6.23), back into (3.6.11) yields

$$\begin{aligned}
&\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2-\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2\\
&+2\left\|\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right)\right\|_0^2\\
&+\frac{\triangle t}{Re}\left(\left\|\triangle(\phi^{n+1}-\phi^{n})\right\|_0^2-\left\|\triangle(\phi^{n}-\phi^{n-1})\right\|_0^2\right)\\
&\le B_2^n+CRe\triangle t\left(\left\|\mathbf{E}^{n}\right\|_0^2+\left\|\mathbf{E}^{n-1}\right\|_0^2\right)\\
&\quad+CRe\triangle t^2\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|p_t(t)\|_0^2\right)dt.
\end{aligned}\tag{3.6.24}$$

Summing over n from 1 to N, and noting that $\mathbf{E}^0=0$ and $\phi^0=0$, implies

$$\begin{aligned}
&\left\|\mathbf{E}^{N+1}-\mathbf{E}^{N}\right\|_0^2+\frac{\triangle t}{Re}\left\|\triangle\left(\phi^{N+1}-\phi^{N}\right)\right\|_0^2\\
&+\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2+2\sum_{n=1}^{N}\left\|\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right\|_0^2\\
&+\frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right)\right\|_0^2\le\sum_{n=1}^{N}B_2^n+\frac{\triangle t}{Re}\left\|\triangle\phi^1\right\|_0^2\\
&\quad+\left\|\mathbf{E}^1\right\|_0^2+CRe\triangle t\sum_{n=0}^{N}\left\|\mathbf{E}^{n}\right\|_0^2\\
&\quad+CRe\triangle t^2\int_0^{t_{N+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|p_t(t)\|_0^2\right)dt.
\end{aligned}\tag{3.6.25}$$

In view of Lemmas 1.5, 1.6, and 3.7, we conclude

$$\begin{aligned}
&\left\|\mathbf{E}^{N+1}-\mathbf{E}^{N}\right\|_0^2+\frac{\triangle t}{Re}\left\|\triangle\left(\phi^{N+1}-\phi^{N}\right)\right\|_0^2\\
&+\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2+2\sum_{n=1}^{N}\left\|\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right\|_0^2\\
&+\frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right)\right\|_0^2\le C\triangle t^2+\sum_{n=1}^{N}B_2^n.
\end{aligned}\tag{3.6.26}$$

It remains to estimate B_2^n. By Lemma 1.4 and Theorem 3.3, we obtain

$$\begin{aligned} B_2^n &\leq C\triangle t\|\mathbf{E}^n - \mathbf{E}^{n-1}\|_0^{\frac{1}{2}}\|\mathbf{E}^n - \mathbf{E}^{n-1}\|_1^{\frac{1}{2}}\left\|\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 \\ &\leq C\triangle t^{\frac{5}{4}}\|\mathbf{E}^n - \mathbf{E}^{n-1}\|_1^{\frac{1}{2}}\left\|\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 \\ &\leq \frac{\triangle t}{4Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\right\|_0^2 + C\triangle t^{\frac{3}{2}}\left\|\widehat{\mathbf{E}}^n\right\|_1^2, \end{aligned} \tag{3.6.27}$$

because of (3.4.7). Since $\triangle t\sum_{n=1}^{N}\left\|\nabla\widehat{\mathbf{E}}^n\right\|_0^2 \leq C\triangle t$, by Theorem 3.3, we have

$$\sum_{n=1}^{N} B_2^n \leq \frac{\triangle t}{4Re}\sum_{n=1}^{N}\left\|\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\right\|_0^2 + C\triangle t^{\frac{3}{2}}. \tag{3.6.28}$$

So (3.6.26) can be bounded by

$$\begin{aligned} &\|\mathbf{E}^{N+1} - \mathbf{E}^N\|_0^2 + \frac{\triangle t}{Re}\|\triangle\left(\phi^{N+1} - \phi^N\right)\|_0^2 \\ &+\sum_{n=1}^{N}\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\|_0^2 + 2\sum_{n=1}^{N}\|\nabla\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right)\|_0^2 \\ &+\frac{\triangle t}{4Re}\sum_{n=1}^{N}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right)\right\|_0^2 \leq C\triangle t^{\frac{3}{2}}. \end{aligned} \tag{3.6.29}$$

Since this is not yet the correct order, we estimate $\mathbf{B}_2^n$ again, but this time employing (3.6.29). We now use the improved estimates

$$\|\mathbf{E}^{n+1} - \mathbf{E}^n\|_0 \leq C\triangle t^{\frac{3}{4}}, \tag{3.6.30}$$

and

$$\|\nabla(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_0 \leq C\triangle t^{\frac{1}{4}} \tag{3.6.31}$$

the latter resulting from (3.4.7), we see that

$$\begin{aligned} B_2^n &\leq C\triangle t\|\mathbf{E}^n - \mathbf{E}^{n-1}\|_0^{\frac{1}{2}}\|\mathbf{E}^n - \mathbf{E}^{n-1}\|_1^{\frac{1}{2}}\left\|\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 \\ &\leq C\triangle t^{\frac{3}{2}}\left\|\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 \\ &\leq \frac{\triangle t}{4Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\right\|_0^2 + C\triangle t^2\left\|\widehat{\mathbf{E}}^n\right\|_1^2. \end{aligned} \tag{3.6.32}$$

Since $\triangle t \sum_{n=1}^{N} \left\| \nabla \widehat{\mathbf{E}}^{n} \right\|_0^2 \leq C \triangle t$, by Theorem 3.3, we realize that (3.6.28) has been improved as follows:

$$\sum_{n=1}^{N} B_2^n \leq \frac{\triangle t}{4Re} \sum_{n=1}^{N} \left\| \nabla (\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^{n}) \right\|_0^2 + C \triangle t^2. \tag{3.6.33}$$

Inserting (3.6.33) into (3.6.26) gives (3.6.9), as asserted. ■

Lemma 3.9 *Suppose Assumptions 1-3 hold. If*

$$\left\| \nabla \mathbf{u}_t(0) \right\|_0 \leq C, \tag{3.6.34}$$

then the error functions of Algorithms 3.1-3.2 satisfy

$$\begin{aligned} \left\| \mathbf{E}^{N+1} - \mathbf{E}^{N} \right\|_{\mathbf{Z}^*}^2 &+ \sum_{n=1}^{N} \left\| \mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} \right\|_{\mathbf{Z}^*}^2 \\ &+ \frac{\triangle t}{Re} \sum_{n=1}^{N} \left\| \mathbf{E}^{n+1} - \mathbf{E}^{n} \right\|_0^2 \leq C \triangle t^3. \end{aligned} \tag{3.6.35}$$

Proof. Let $(\mathbf{v}^{n+1}, q^{n+1}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be the solution of the Stokes equation (3.4.22). Multiplying (3.6.10) by $2\triangle t(\mathbf{v}^{n+1} - \mathbf{v}^{n}) \in \mathbf{H}_0^1(\Omega)$, we get

$$\begin{aligned} & 2 \left\langle \mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} \,,\, \mathbf{v}^{n+1} - \mathbf{v}^{n} \right\rangle \\ & \qquad + \frac{2\triangle t}{Re} \left\langle \nabla (\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^{n}) \,,\, \nabla (\mathbf{v}^{n+1} - \mathbf{v}^{n}) \right\rangle \\ & = 2\triangle t \left\langle \mathbf{R}_{n+1} - \mathbf{R}_{n} \,,\, \mathbf{v}^{n+1} - \mathbf{v}^{n} \right\rangle \\ & \quad - 2\triangle t \mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\ & \quad + 2\triangle t \mathcal{N}(\mathbf{u}^{n}, \widehat{\mathbf{u}}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\ & \quad + 2\triangle t \mathcal{N}(\mathbf{u}(t_{n}), \mathbf{u}(t_{n}), \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\ & \quad - 2\triangle t \mathcal{N}(\mathbf{u}^{n-1}, \widehat{\mathbf{u}}^{n}, \mathbf{v}^{n+1} - \mathbf{v}^{n}). \end{aligned} \tag{3.6.36}$$

With the aid of (3.4.22), the left hand side can be handled as follows

$$
\begin{aligned}
&2\left\langle \mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1}\,,\, \mathbf{v}^{n+1} - \mathbf{v}^{n} \right\rangle \\
&= 2\left\langle \nabla(\mathbf{v}^{n+1} - 2\mathbf{v}^{n} + \mathbf{v}^{n-1})\,,\, \nabla(\mathbf{v}^{n+1} - \mathbf{v}^{n}) \right\rangle \\
&= \left\| \nabla(\mathbf{v}^{n+1} - \mathbf{v}^{n}) \right\|_0^2 - \left\| \nabla(\mathbf{v}^{n} - \mathbf{v}^{n-1}) \right\|_0^2 \\
&\quad + \left\| \nabla(\mathbf{v}^{n+1} - 2\mathbf{v}^{n} + \mathbf{v}^{n-1}) \right\|_0^2,
\end{aligned}
\tag{3.6.37}
$$

and use (3.4.4)

$$
\begin{aligned}
&\frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^{n})\,,\, \nabla(\mathbf{v}^{n+1} - \mathbf{v}^{n}) \right\rangle \\
&= \frac{2\triangle t}{Re}\left\langle (\mathbf{E}^{n+1} - \mathbf{E}^{n}) - \nabla(q^{n+1} - q^{n})\,,\, \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^{n} \right\rangle \\
&= \frac{2\triangle t}{Re}\left\| \mathbf{E}^{n+1} - \mathbf{E}^{n} \right\|_0^2 - \frac{2\triangle t}{Re}\left\langle \nabla(q^{n+1} - q^{n})\,,\, \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^{n} \right\rangle.
\end{aligned}
\tag{3.6.38}
$$

In view of Lemma 3.5, (3.6.36) becomes

$$
\begin{aligned}
&\left\| \nabla(\mathbf{v}^{n+1} - \mathbf{v}^{n}) \right\|_0^2 - \left\| \nabla(\mathbf{v}^{n} - \mathbf{v}^{n-1}) \right\|_0^2 \\
&+ \left\| \nabla(\mathbf{v}^{n+1} - 2\mathbf{v}^{n} + \mathbf{v}^{n-1}) \right\|_0^2 + \frac{2\triangle t}{Re}\left\| \mathbf{E}^{n+1} - \mathbf{E}^{n} \right\|_0^2 \\
&= \frac{2\triangle t}{Re}\left\langle \nabla(q^{n+1} - q^{n})\,,\, \nabla(\phi^{n+1} - 2\phi^{n} + \phi^{n-1}) \right\rangle \\
&\quad + 2\triangle t \left\langle \mathbf{R}_{n+1} - \mathbf{R}_{n}\,,\, \mathbf{v}^{n+1} - \mathbf{v}^{n} \right\rangle \\
&\quad - 2\triangle t \quad \Big(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\
&\qquad - \mathcal{N}(\mathbf{u}(t_{n}), \mathbf{u}(t_{n}), \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\
&\qquad - \mathcal{N}(\mathbf{u}^{n}, \widehat{\mathbf{u}}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^{n}) \\
&\qquad + \mathcal{N}(\mathbf{u}^{n-1}, \widehat{\mathbf{u}}^{n}, \mathbf{v}^{n+1} - \mathbf{v}^{n}) \Big) \\
&= A_1 + A_2 + A_3.
\end{aligned}
\tag{3.6.39}
$$

We estimate each term A_i separately. We first use Assumption 1 to write

$$\begin{aligned} A_1 &\le \frac{C\triangle t}{Re}\big\|\nabla(q^{n+1}-q^n)\big\|_0\big\|\nabla(\phi^{n+1}-2\phi^n+\phi^{n-1})\big\|_0 \\ &\le \frac{\triangle t}{2Re}\big\|\mathbf{E}^{n+1}-\mathbf{E}^n\big\|_0^2+\frac{C\triangle t}{Re}\big\|\nabla(\phi^{n+1}-2\phi^n+\phi^{n-1})\big\|_0^2. \end{aligned} \tag{3.6.40}$$

Since

$$\begin{aligned} &2\triangle t\,\langle \mathbf{R}_n\,,\,\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1}\rangle \\ &\quad\le C\triangle t^2\|\mathbf{R}_n\|_{\mathbf{Z}^*}^2+\frac{1}{2}\big\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1})\big\|_0^2 \\ &\quad\le C\triangle t^3\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2\,dt+\frac{1}{2}\big\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1})\big\|_0^2, \end{aligned} \tag{3.6.41}$$

we deduce

$$\begin{aligned} A_2 &= 2\triangle t\,\big\langle \mathbf{R}_{n+1}\,,\,\mathbf{v}^{n+1}-\mathbf{v}^n\big\rangle-2\triangle t\,\big\langle \mathbf{R}_n\,,\,\mathbf{v}^n-\mathbf{v}^{n-1}\big\rangle \\ &\quad -2\triangle t\,\big\langle \mathbf{R}_n\,,\,\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1}\big\rangle \\ &\le 2\triangle t\,\big\langle \mathbf{R}_{n+1}\,,\,\mathbf{v}^{n+1}-\mathbf{v}^n\big\rangle-2\triangle t\,\big\langle \mathbf{R}_n\,,\,\mathbf{v}^n-\mathbf{v}^{n-1}\big\rangle \\ &\quad +C\triangle t^3\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2\,dt+\frac{1}{2}\big\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1})\big\|_0^2. \end{aligned} \tag{3.6.42}$$

The convection A_3 can be split as follows

$$\begin{aligned} A_3 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{v}^{n+1}-\mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n),\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{v}^{n+1}-\mathbf{v}^n) \\ &\quad +2\triangle t\mathcal{N}(\mathbf{u}^n-\mathbf{u}^{n-1},\widehat{\mathbf{u}}^{n+1},\mathbf{v}^{n+1}-\mathbf{v}^n) \\ &\quad +2\triangle t\mathcal{N}(\mathbf{u}^{n-1},\widehat{\mathbf{u}}^{n+1}-\widehat{\mathbf{u}}^n,\mathbf{v}^{n+1}-\mathbf{v}^n), \end{aligned} \tag{3.6.43}$$

whence

$$\begin{aligned} A_3 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}) - 2\mathbf{u}(t_n) + \mathbf{u}(t_{n-1}), \mathbf{u}(t_{n+1}), \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{E}^n - \mathbf{E}^{n-1}, \mathbf{u}(t_{n+1}), \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}^n - \mathbf{u}^{n-1}, \widehat{\mathbf{E}}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1}), \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{E}^{n-1}, \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}^{n-1}, \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n, \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &= A_{3,1} + A_{3,2} + A_{3,3} + A_{3,4} + A_{3,5} + A_{3,6}. \end{aligned} \tag{3.6.44}$$

Using Lemma 1.4, the first two terms become

$$\begin{aligned} A_{3,1} &\leq C\triangle t\|\mathbf{u}(t_{n+1}) - 2\mathbf{u}(t_n) + \mathbf{u}(t_{n-1})\|_0\|\mathbf{u}(t_{n+1})\|_2\big\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\big\|_0 \\ &\leq C\triangle t^4\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2 dt + C\triangle t\big\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\big\|_0^2, \end{aligned} \tag{3.6.45}$$

and

$$\begin{aligned} A_{3,2} &\leq C\triangle t\big\|\mathbf{E}^n - \mathbf{E}^{n-1}\big\|_0\|\mathbf{u}(t_{n+1})\|_2\big\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\big\|_0 \\ &\leq \frac{\triangle t}{12Re}\left(\big\|\mathbf{E}^{n+1} - \mathbf{E}^n\big\|_0^2 + \big\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\big\|_0^2\right) \\ &\quad + CRe\triangle t\big\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\big\|_0^2. \end{aligned} \tag{3.6.46}$$

We split the next term as follows:

$$\begin{aligned} A_{3,3} &= 2\triangle t\mathcal{N}(\mathbf{E}^n - \mathbf{E}^{n-1}, \widehat{\mathbf{E}}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1}), \widehat{\mathbf{E}}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &= A_{3,3,1} + A_{3,3,2}. \end{aligned} \tag{3.6.47}$$

Since $\left\|\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0 \le C\triangle t$ by Lemma 3.8, we have from Lemmas 1.3 and 1.4

$$
\begin{aligned}
A_{3,3,1} &\le C\triangle t\left\|\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\le C\triangle t^2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \\
&\le CRe\triangle t^3\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\triangle t}{12Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0.
\end{aligned}
\tag{3.6.48}
$$

Since $\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0 \le \triangle t^{\frac{1}{2}}$ by Lemma 3.3, again Lemmas 1.3 and 1.4 yield

$$
\begin{aligned}
A_{3,3,2} &\le C\triangle t\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\le C\triangle t^{\frac{3}{2}}\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \\
&\le CRe\triangle t^3\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_1^2 dt + \frac{\triangle t}{12Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2.
\end{aligned}
\tag{3.6.49}
$$

Since $\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1 \le C\triangle t^{\frac{1}{2}}$, we have

$$
\begin{aligned}
A_{3,4} &\le C\triangle t\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_1 \\
&\le C\triangle t^{\frac{3}{2}}\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_1 \\
&\le C\triangle t^3\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_1^2 dt + C\triangle t\left\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n)\right\|_0^2.
\end{aligned}
\tag{3.6.50}
$$

Theorem 3.3 yields $\left\|\mathbf{E}^{n-1}\right\|_0 \le \triangle t^{\frac{1}{2}}$, and this in turn implies

$$
\begin{aligned}
A_{3,5} &\le C\triangle t\left\|\mathbf{E}^{n-1}\right\|_0\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\le C\triangle t^{\frac{3}{2}}\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \\
&\le CRe\triangle t^3\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_1^2 dt + \frac{\triangle t}{12Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2.
\end{aligned}
\tag{3.6.51}
$$

We split $A_{3,6}$ as follows:

$$
\begin{aligned}
A_{3,6} &= 2\triangle t\mathcal{N}(\mathbf{E}^{n-1}, \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n, \mathbf{v}^{n+1} - \mathbf{v}^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1}), \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n, \mathbf{v}^{n+1} - \mathbf{v}^n) = A_{3,6,1} + A_{3,6,2}.
\end{aligned}
\tag{3.6.52}
$$

Lemma 1.4 yields

$$\begin{aligned} A_{3,6,1} &\leq C\triangle t\left\|\mathbf{E}^{n-1}\right\|_0\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_1\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2 \\ &\leq C\triangle t^{\frac{3}{2}}\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_1\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0 \\ &\leq CRe\triangle t^2\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\frac{\triangle t}{12Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2. \end{aligned} \tag{3.6.53}$$

Lemmas 1.3 and 1.4 give

$$\begin{aligned} A_{3,6,2} &\leq C\triangle t\|\mathbf{u}(t_{n-1})\|_2\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_0\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_1 \\ &\leq \frac{\triangle t}{12Re}\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_0^2+CRe\triangle t\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_1^2. \end{aligned} \tag{3.6.54}$$

Collecting (3.6.45)-(3.6.54), we see that

$$\begin{aligned} A_3 \leq\ & \frac{\triangle t}{2Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+C(1+Re)\triangle t\left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\|_0^2 \\ &+\frac{\triangle t}{12Re}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2 \\ &+CRe\triangle t^3\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+CRe\triangle t^2\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2 \\ &+C(1+Re)\triangle t^3\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2\right)dt. \end{aligned} \tag{3.6.55}$$

Inserting (3.6.40), (3.6.42), and (3.6.55) into (3.6.39), leads to

$$\begin{aligned} &\left\|\nabla\left(\mathbf{v}^{n+1}-\mathbf{v}^n\right)\right\|_0^2-\left\|\nabla(\mathbf{v}^n-\mathbf{v}^{n-1})\right\|_0^2+\frac{\triangle t}{Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 \\ &+\frac{1}{2}\left\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^n+\mathbf{v}^{n-1})\right\|_0^2\leq C\triangle t\left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\|_0^2 \\ &\quad+\frac{C\triangle t}{Re}\left(\left\|\nabla(\phi^{n+1}-2\phi^n+\phi^{n-1})\right\|_0^2+\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2\right) \\ &\quad+CRe\triangle t^3\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+CRe\triangle t^2\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2 \\ &\quad+C(1+Re)\triangle t^3\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2\right)dt \\ &\quad+2\triangle t\left\langle\mathbf{R}_{n+1}\,,\,\mathbf{v}^{n+1}-\mathbf{v}^n\right\rangle-2\triangle t\left\langle\mathbf{R}_n\,,\,\mathbf{v}^n-\mathbf{v}^{n-1}\right\rangle. \end{aligned} \tag{3.6.56}$$

Summing over n from 1 to N,

$$
\begin{aligned}
&\left\|\nabla\left(\mathbf{v}^{N+1}-\mathbf{v}^{N}\right)\right\|_0^2+\frac{1}{2}\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^{n}+\mathbf{v}^{n-1})\right\|_0^2\\
&+\frac{\triangle t}{Re}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2\leq C\triangle t\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^{n})\right\|_0^2\\
&\quad+\frac{C\triangle t}{Re}\sum_{n=1}^{N}\left(\left\|\nabla(\phi^{n+1}-2\phi^{n}+\phi^{n-1})\right\|_0^2+\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2\right)\\
&\quad+CRe\triangle t^3\sum_{n=1}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+CRe\triangle t^2\sum_{n=1}^{N}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2\\
&\quad+C(1+Re)\triangle t^3\int_0^{t_{N+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2\right)dt\\
&\quad+\left\|\nabla\mathbf{v}^1\right\|_0^2+2\triangle t\left\langle\mathbf{R}_{N+1}\,,\,\mathbf{v}^{N+1}-\mathbf{v}^{N}\right\rangle-2\triangle t\left\langle\mathbb{R}_1\,,\,\mathbf{v}^1-\mathbf{v}^0\right\rangle.
\end{aligned}
\tag{3.6.57}
$$

The residual terms become

$$
\begin{aligned}
2\triangle t\left\langle\mathbf{R}_{N+1}\,,\,\mathbf{v}^{N+1}-\mathbf{v}^{N}\right\rangle\ \leq\ & C\triangle t^3\int_{t_N}^{t_{N+1}}\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2dt\\
&+\frac{1}{2}\left\|\nabla(\mathbf{v}^{N+1}-\mathbf{v}^{N})\right\|_0^2,
\end{aligned}
\tag{3.6.58}
$$

and

$$
-2\triangle t\left\langle\mathbb{R}_1\,,\,\mathbf{v}^1-\mathbf{v}^0\right\rangle\leq C\triangle t^3\int_0^{t_1}\|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2dt+\left\|\mathbf{v}^1\right\|_0^2,
\tag{3.6.59}
$$

because $\mathbf{v}^0=0$. Lemmas 1.5-1.7, and 3.7-3.8, together with Theorems 3.3-3.4, imply

$$
\begin{aligned}
&\left\|\nabla\left(\mathbf{v}^{N+1}-\mathbf{v}^{N}\right)\right\|_0^2+\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}^{n+1}-2\mathbf{v}^{n}+\mathbf{v}^{n-1})\right\|_0^2\\
&+\frac{\triangle t}{Re}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2\leq C\triangle t\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}^{n+1}-\mathbf{v}^{n})\right\|_0^2+C\triangle t^3.
\end{aligned}
\tag{3.6.60}
$$

Finally, the discrete Gronwall lemma gives (3.6.35), as desired. ∎

Now we estimate the error e^{n+1} of pressure of the Algorithms 3.1-3.2

Theorem 3.9 *Suppose Assumptions 1-3 hold. If*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le C, \tag{3.6.61}$$

then the pressure error functions e^{n+1} *of Algorithms 3.1-3.2 satisfies*

$$\triangle t \sum_{n=0}^{N} \left\|e^{n+1}\right\|_0^2 \le C\triangle t. \tag{3.6.62}$$

PROOF. By definition (3.6.1) of discrete pressure p^{n+1}, (3.4.10) can be rewritten by for all $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned}
\left\langle e^{n+1}, \operatorname{div} \mathbf{w}\right\rangle = & \left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}, \mathbf{w}\right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{w}\right\rangle \\
& + \mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w}) - \mathcal{N}(\mathbf{u}^n, \widehat{\mathbf{u}}^{n+1}, \mathbf{w}) \qquad (3.6.63)\\
& - \frac{1}{Re}\left\langle \triangle(\phi^{n+1}-\phi^n), \operatorname{div} \mathbf{w}\right\rangle - \left\langle \mathbf{R}_{n+1}, \mathbf{w}\right\rangle .
\end{aligned}$$

By continuous inf-sup condition, there exists $\mathbf{z}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and $\beta > 0$ such that

$$\left\langle e^{n+1}, \operatorname{div} \mathbf{z}^{n+1}\right\rangle = \|e^{n+1}\|_0^2 \quad \text{and } \left\|\mathbf{z}^{n+1}\right\|_1 \le \frac{1}{\beta}\left\|e^{n+1}\right\|_0. \tag{3.6.64}$$

So

$$\begin{aligned}
\|e^{n+1}\|_0^2 &= \left\langle e^{n+1}, \operatorname{div} \mathbf{z}^{n+1}\right\rangle \\
&= \left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}, \mathbf{z}^{n+1}\right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{z}^{n+1}\right\rangle \\
&\quad + \left(\mathcal{N}\left(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{z}^{n+1}\right) - \mathcal{N}\left(\mathbf{u}^n, \widehat{\mathbf{u}}^{n+1}, \mathbf{z}^{n+1}\right)\right) \qquad (3.6.65)\\
&\quad - \frac{1}{Re}\left\langle \triangle(\phi^{n+1}-\phi^n), \operatorname{div} \mathbf{z}^{n+1}\right\rangle - \left\langle \mathbf{R}_{n+1}, \mathbf{z}^{n+1}\right\rangle \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

We now proved to estimate each term A_i separately. We first note that

$$A_1 \le \frac{\|\mathbf{E}^{n+1}-\mathbf{E}^n\|_0\|\mathbf{z}^{n+1}\|_0}{\triangle t} \le \frac{C}{\beta\triangle t}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0\left\|e^{n+1}\right\|_0, \tag{3.6.66}$$

and

$$A_2 \le \frac{C}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\|\nabla\mathbf{z}^{n+1}\|_0 \le \frac{C}{\beta Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\|e^{n+1}\|_0. \tag{3.6.67}$$

The convection term A_3 can split as follows:

$$\begin{aligned} A_3 &= -\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{z}^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\mathbf{z}^{n+1})-\mathcal{N}(\mathbf{u}^n,\widehat{\mathbf{E}}^{n+1},\mathbf{z}^{n+1}) \\ &= -\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{z}^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{E}^n,\mathbf{u}(t_{n+1}),\mathbf{z}^{n+1})+\mathcal{N}(\mathbf{E}^n,\widehat{\mathbf{E}}^{n+1},\mathbf{z}^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{u}(t_n),\widehat{\mathbf{E}}^{n+1},\mathbf{z}^{n+1}) = A_{3,1}+A_{3,2}+A_{3,3}+A_{3,4}. \end{aligned} \tag{3.6.68}$$

In view of Lemma 1.4, we have

$$\begin{aligned} A_{3,1} &\le C\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{z}^{n+1}\right\|_0 \\ &\le \frac{C}{\beta}\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\left\|e^{n+1}\right\|_0, \end{aligned} \tag{3.6.69}$$

and

$$A_{3,2} \le C\|\mathbf{E}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{z}^{n+1}\right\|_0 \le \frac{C}{\beta}\|\mathbf{E}^n\|_0\left\|e^{n+1}\right\|_0. \tag{3.6.70}$$

Since $\|\mathbf{E}^n\|_1 \le C$ by Theorem 3.3, applying Lemma 1.4, we get

$$A_{3,3} \le \|\mathbf{E}^n\|_1\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1\left\|\mathbf{z}^{n+1}\right\|_1 \le \frac{C}{\beta}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|e^{n+1}\right\|_0, \tag{3.6.71}$$

and invoking also Lemma 1.3 with $\mathbf{u}=\mathbf{u}(t_n)$, we see that

$$A_{3,4} \le \|\mathbf{u}(t_n)\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{z}^{n+1}\right\|_1 \le \frac{C}{\beta}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|e^{n+1}\right\|_0. \tag{3.6.72}$$

Poincare inequality yields $\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0 \le C\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0$, whence

$$A_3 \le \frac{C}{\beta}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0+\|\mathbf{E}^n\|_0+\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\right)\left\|e^{n+1}\right\|_0. \tag{3.6.73}$$

On the other hand, we have

$$\begin{aligned} A_4 &\leq \frac{1}{Re}\left\|\triangle(\phi^{n+1}-\phi^n)\right\|_0\left\|\text{div }\mathbf{z}^{n+1}\right\|_0 \\ &\leq \frac{C}{\beta Re}\left\|\triangle(\phi^{n+1}-\phi^n)\right\|_0\left\|e^{n+1}\right\|_0, \end{aligned} \tag{3.6.74}$$

and

$$A_5 \leq \|\mathbf{R}_{n+1}\|_{-1}\left\|\nabla\mathbf{z}^{n+1}\right\|_0 \leq \frac{C}{\beta}\|\mathbf{R}_{n+1}\|_{-1}\left\|e^{n+1}\right\|_0. \tag{3.6.75}$$

By (3.6.66)-(3.6.67) and (3.6.73)-(3.6.75),

$$\begin{aligned} \|e^{n+1}\|_0 \leq\ & \frac{C}{\beta\triangle t}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0+\frac{C}{\beta}\|\mathbf{R}_{n+1}\|_{-1} \\ &+\frac{C}{\beta Re}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0+\left\|\triangle\left(\phi^{n+1}-\phi^n\right)\right\|_0\right) \\ &+\frac{C}{\beta}\left(\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0+\|\mathbf{E}^n\|_0+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\right). \end{aligned} \tag{3.6.76}$$

Squaring, multiplying by $\triangle t$, and summing over n from 0 to N, we end up with

$$\begin{aligned} \triangle t\sum_{n=0}^{N}\left\|e^{n+1}\right\|_0^2 \leq\ & \frac{C}{\beta^2\triangle t}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 \\ &+\frac{C\triangle t}{\beta^2}\sum_{n=0}^{N}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\|\mathbf{E}^n\|_0^2\right) \\ &+\frac{C\triangle t}{\beta^2 Re^2}\sum_{n=0}^{N}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\triangle\left(\phi^{n+1}-\phi^n\right)\right\|_0^2\right) \\ &+\frac{C\triangle t}{\beta^2}\int_0^{t_{n+1}}\sigma(t)\|\mathbf{u}_{tt}(t)\|_{-1}^2dt+\frac{C\triangle t^2}{\beta^2}\int_0^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt. \end{aligned} \tag{3.6.77}$$

By Theorem 3.4 and Lemmas 3.8-3.9, we finally get (3.6.62). ■

	Neumann	Dirichlet
div	**I** $\left\|\mathbf{E}^{N+1}\right\|_0^2 + \triangle t \left\|e^{N+1}\right\|_0^2 \leq C\triangle t^2$ No Super Convergence of $\dfrac{\partial \phi^{n+1}}{\partial \tau}$ in 3d	**III** $\left\|\mathbf{E}^{N+1}\right\|_0^2 \leq C\triangle t$ **No Pressure**
curl	**II** $\left\|\mathbf{E}^{N+1}\right\|_0^2 + \triangle t \left\|e^{N+1}\right\|_0^2 \leq C\triangle t^2$ **Work on 2D**	**IV** $\left\|\mathbf{E}^{N+1}\right\|_0^2 \leq C\triangle t$ **No Pressure** **Work on 2D**

Table 3.1: Summary of Gauge Methods

3.7 Conclusion and Numerical Results for Gauge methods

The Table 3.1 is the theoretical summary of gauge methods, and the difficulty for boundary differentiation $\dfrac{\partial \phi^{n+1}}{\partial \tau}$ in 3d for Algorithm 3.1 is explained in Figure 4.3 in Section Figure 4.1.

We report here numerical experiments with Algorithms 3.1-3.4 and examples 1.3.1 for several finite element spaces for velocity, pressure, and gauge variable as follows:

$$P_1 - P_1 - P_1, \quad P_1 - P_1 - P_2, \tag{3.7.1}$$

and

$$P_2 - P_1 - P_1, \quad P_2 - P_1 - P_2. \tag{3.7.2}$$

All finite elements are continuous, and the pair $P_2 - P_1$ is the well known stable Taylor-Hood combination. Our aim is to compare their performance for both velocity and pressure in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^\infty(\Omega)$, and several combinations of finite element space. The goal of these experiments are to discover the relations among each discretized space.

3.7.1 Algorithms 3.1 and 3.2 : Neumann Boundary Condition

The combination of (3.7.1) do not satisfy the discrete inf-sup, and that Algorithms 3.1-3.2 do not necessarily converge to the solution of NSE (1.1.1). But, in experiment, the finite element spaces $P_1 - P_1 - P_2$ and $P_2 - P_1 - P_2$ seem to be stable for pressure for both $h = \triangle t^2$ and $h = \triangle t$ (see Figures 3.7, 3.10, 3.19, and 3.22), and the errors for pressure in L^∞ do not decay in other finite element space except Algorithms 3.2 in space $P_1 - P_1 - P_1$. These results do not consist with the concept of inf-sup condition. So we invest the space discetization for gauge methods in Chapter 4.

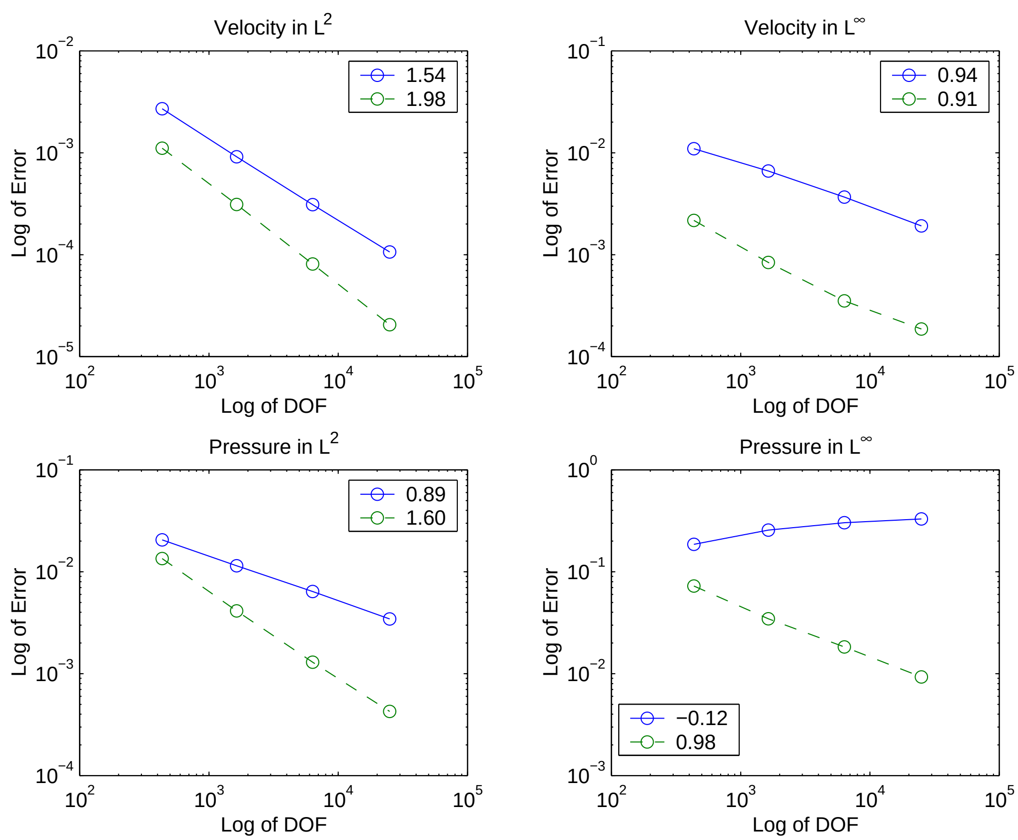


Figure 3.1: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements.

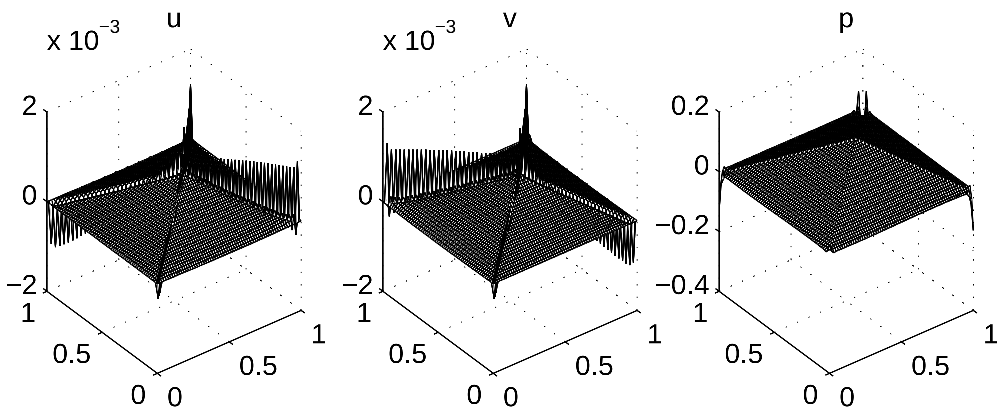


Figure 3.2: Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

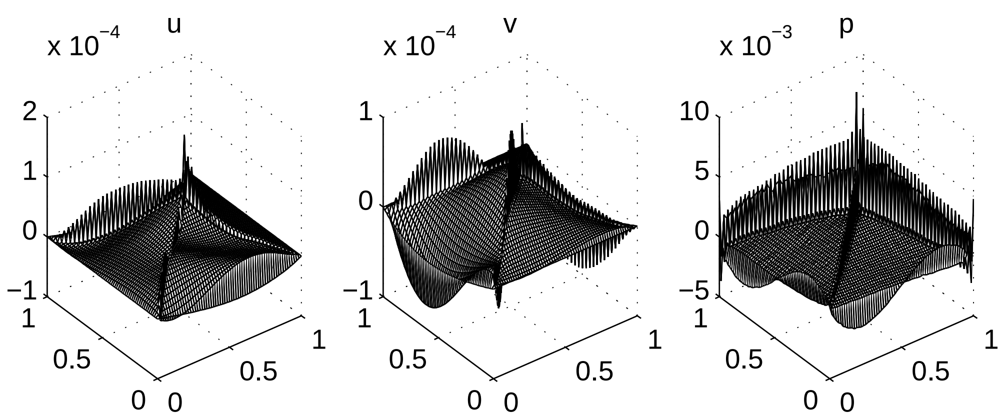


Figure 3.3: Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

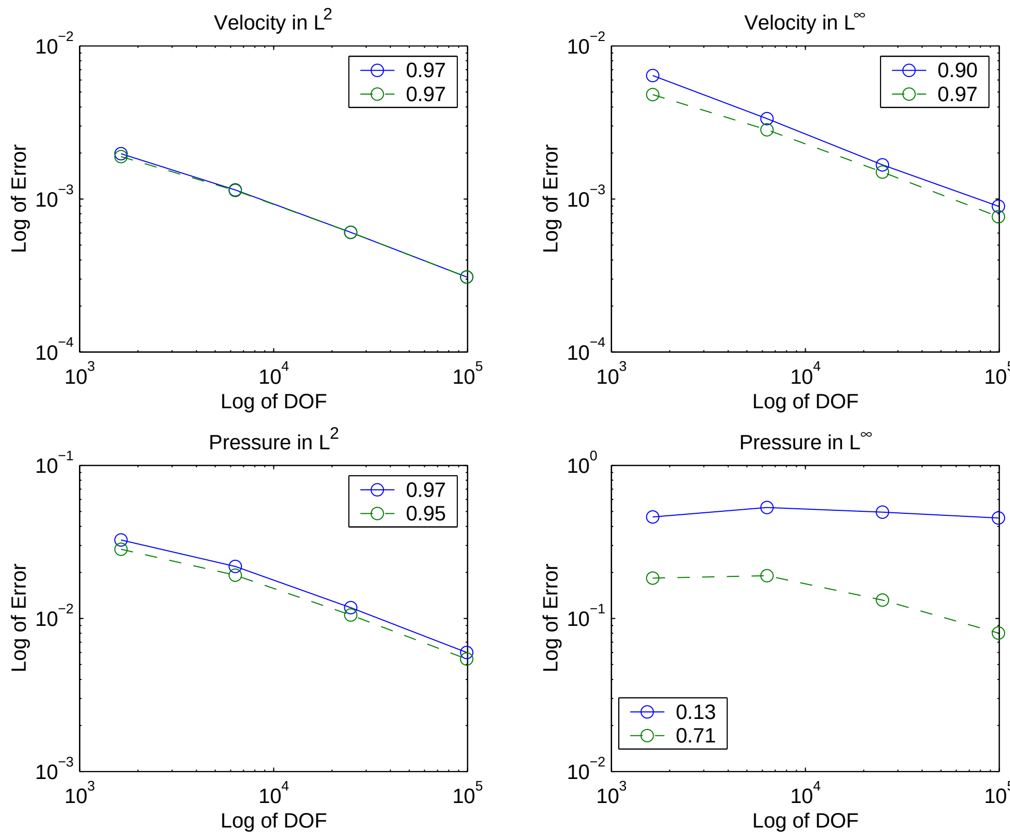


Figure 3.4: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements.

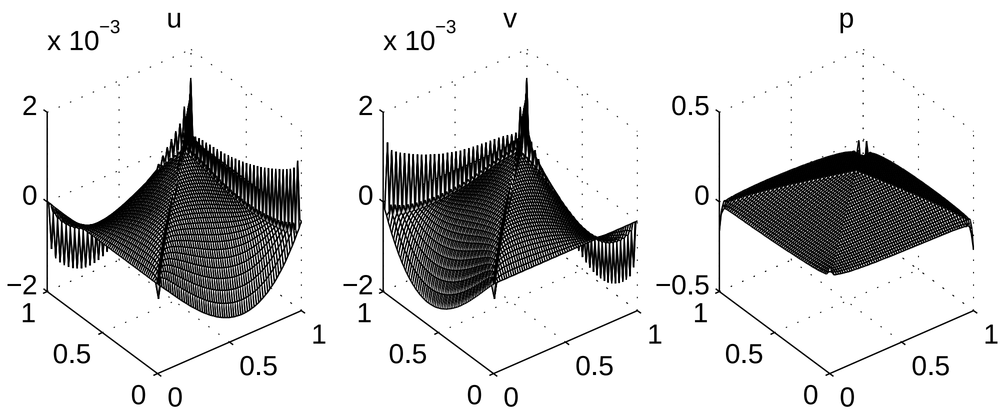


Figure 3.5: Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

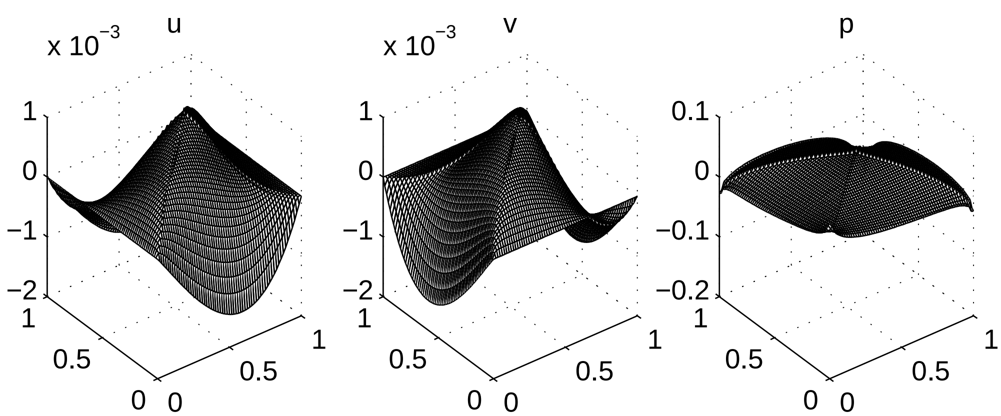


Figure 3.6: Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

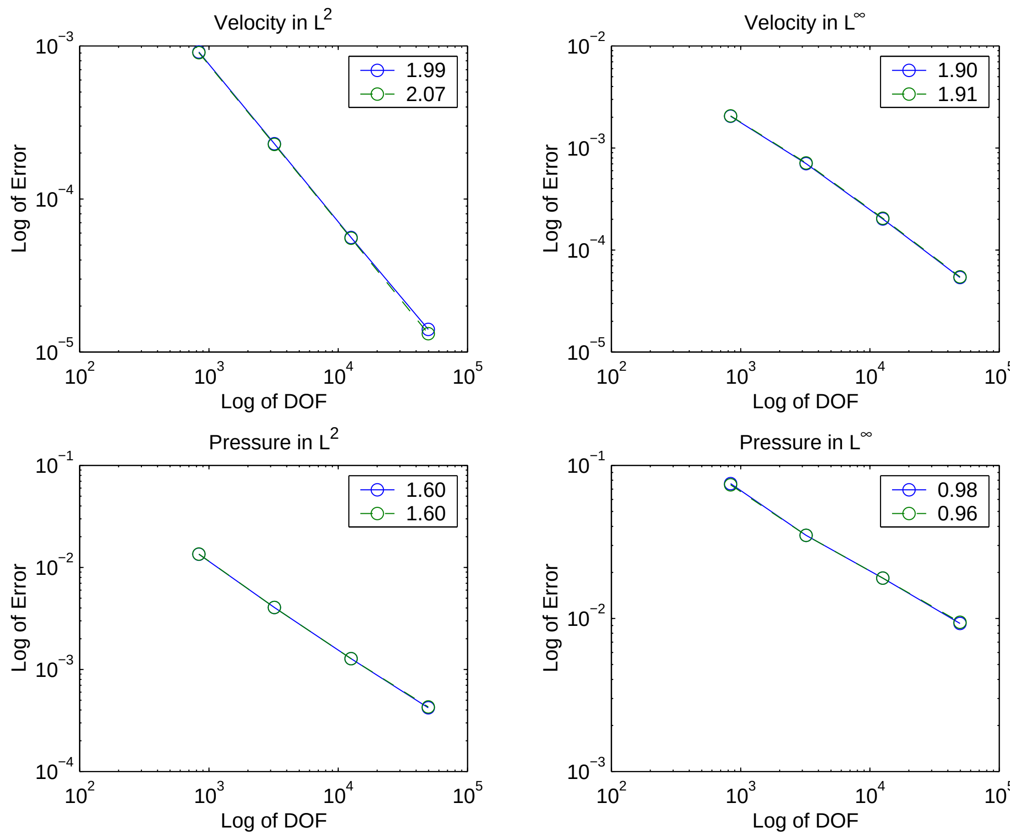


Figure 3.7: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements.

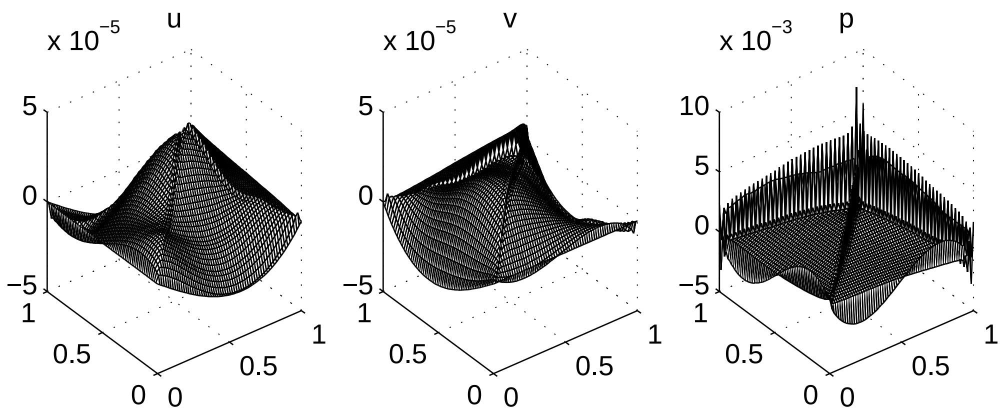


Figure 3.8: Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

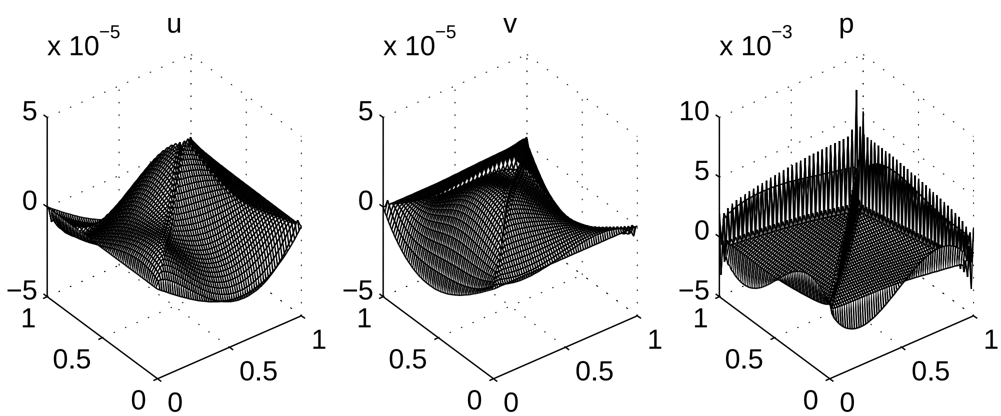


Figure 3.9: Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

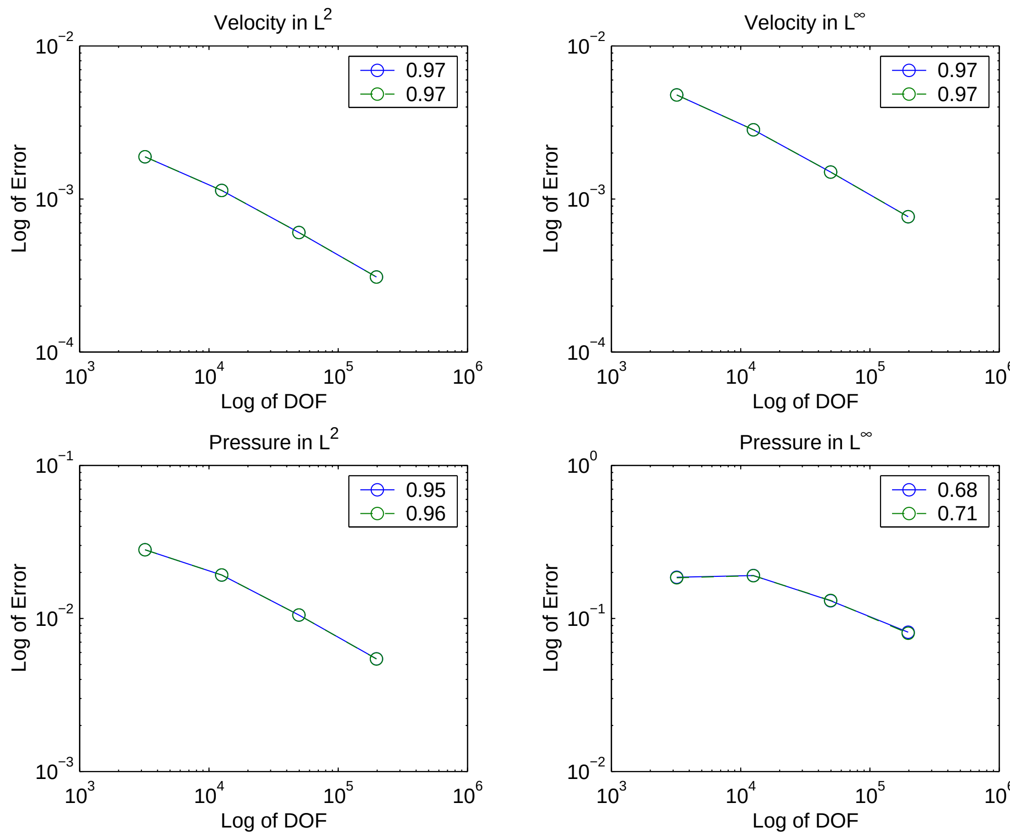


Figure 3.10: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements.

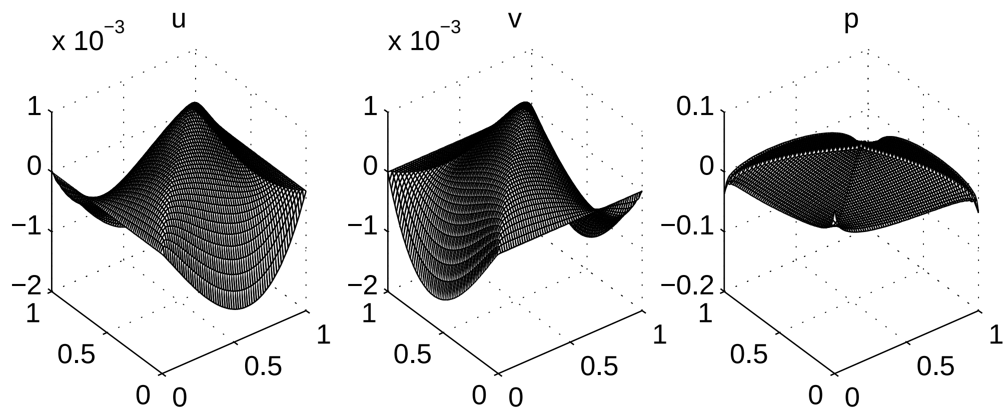


Figure 3.11: Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

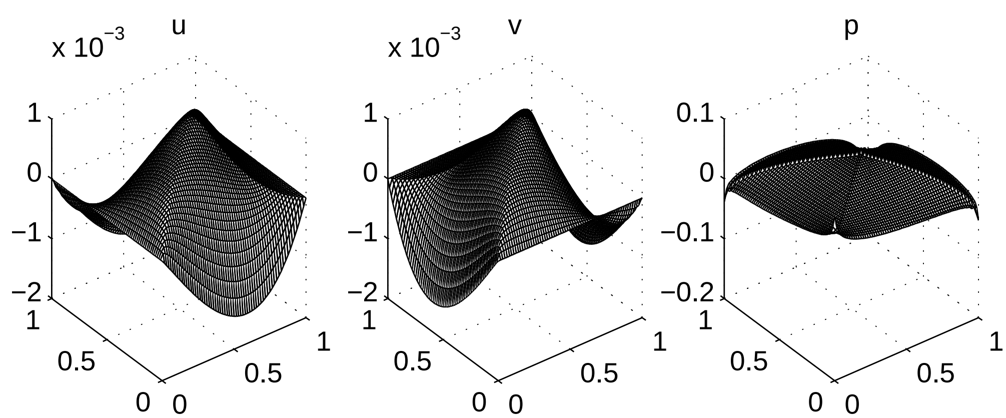


Figure 3.12: Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

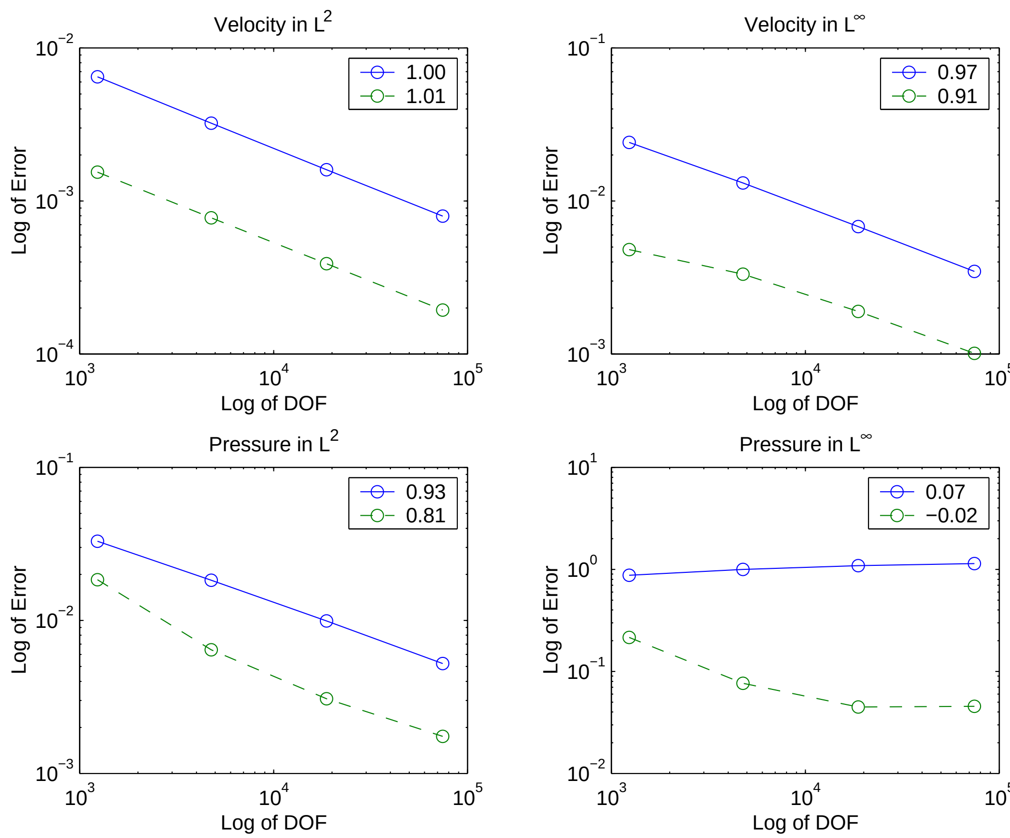


Figure 3.13: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements.

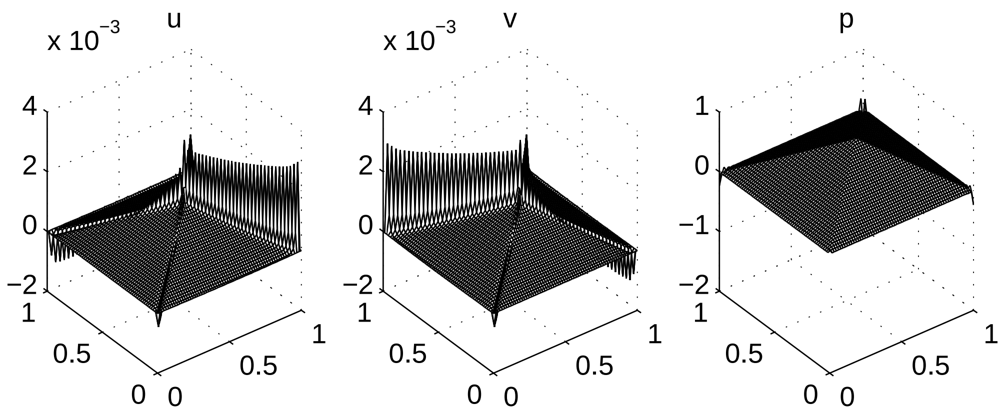


Figure 3.14: Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

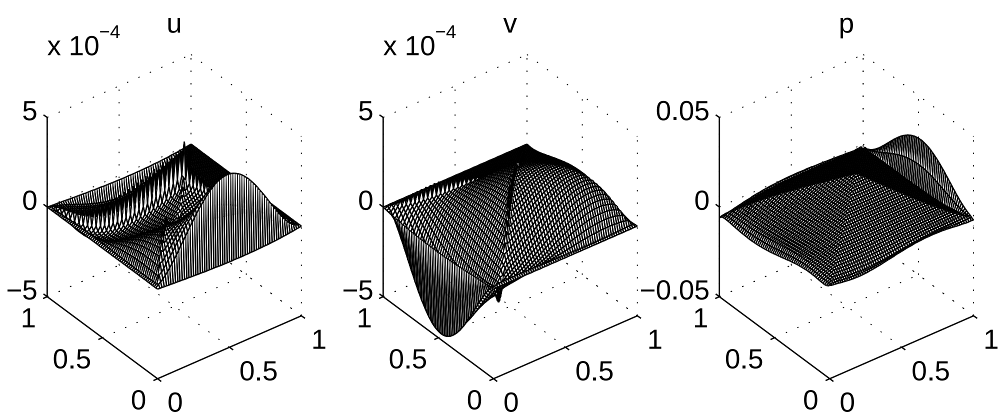


Figure 3.15: Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

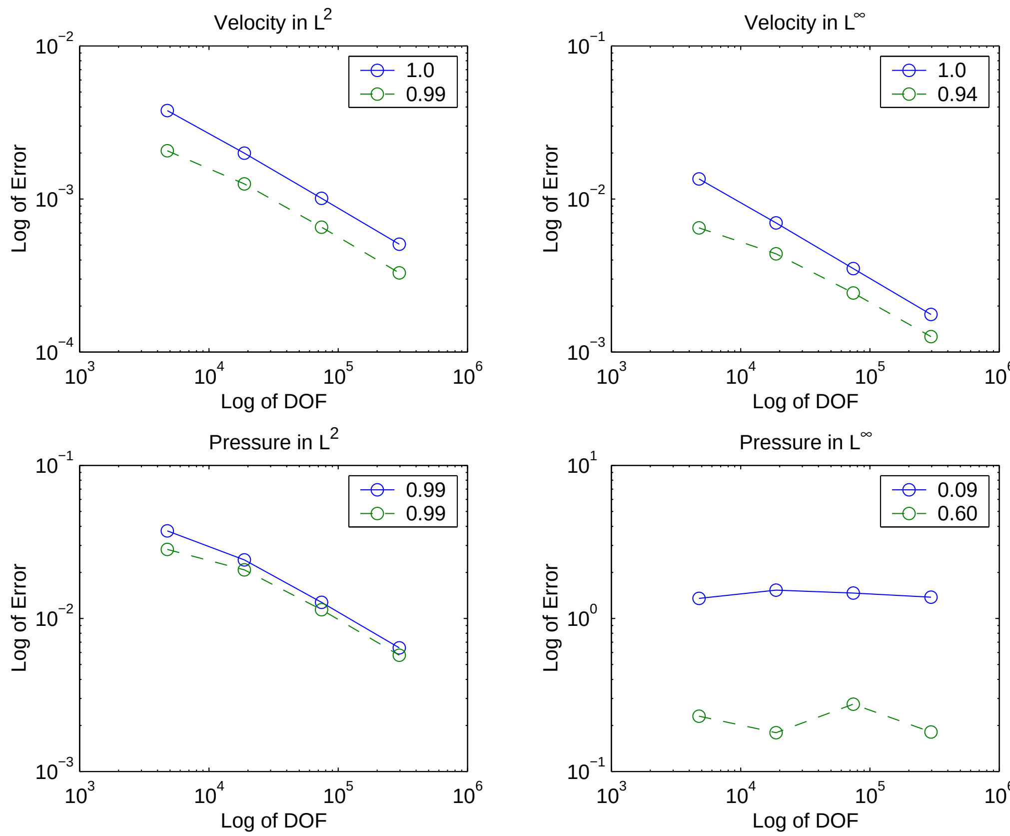


Figure 3.16: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements.

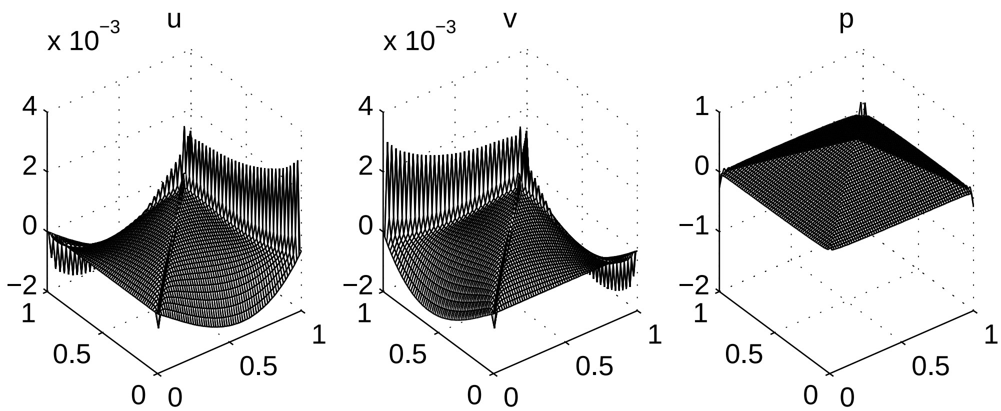


Figure 3.17: Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ (DOF = 74,371) Elements.

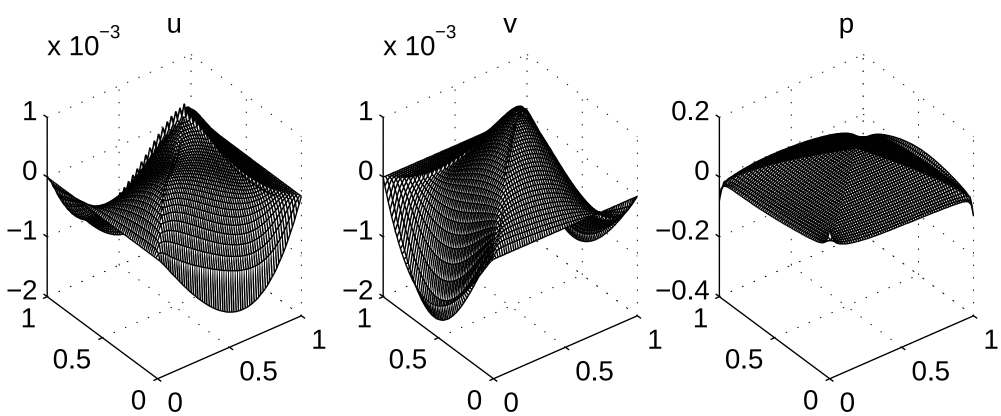


Figure 3.18: Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

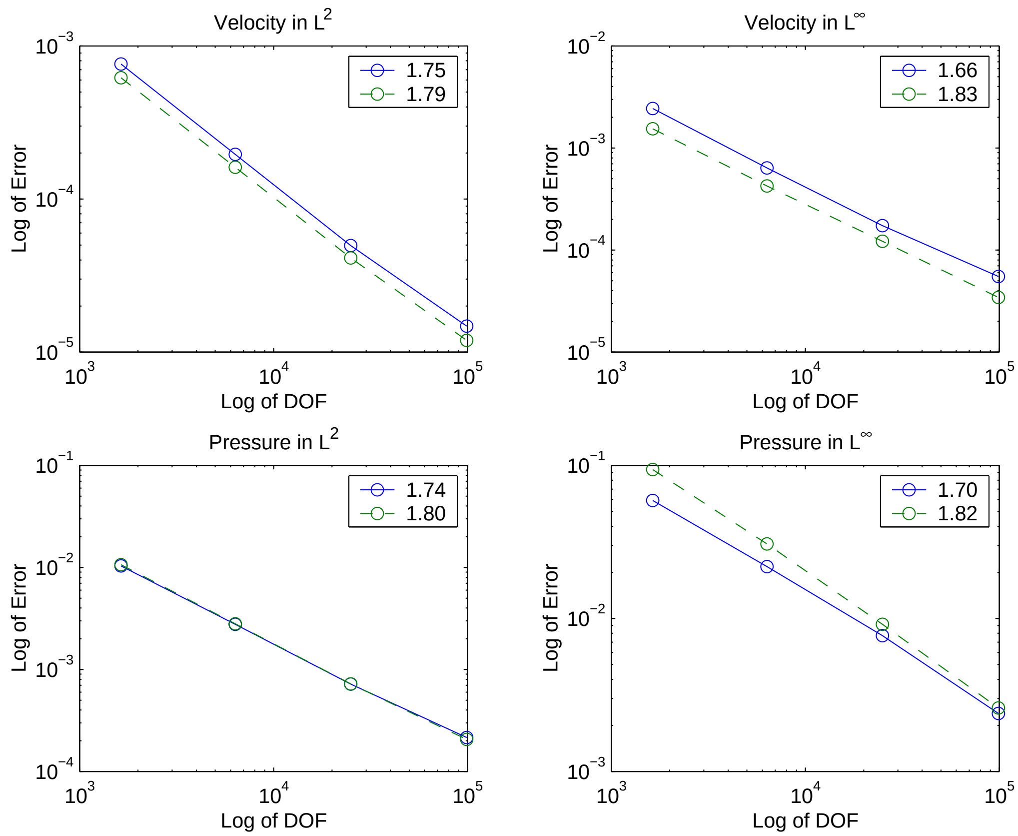


Figure 3.19: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements.

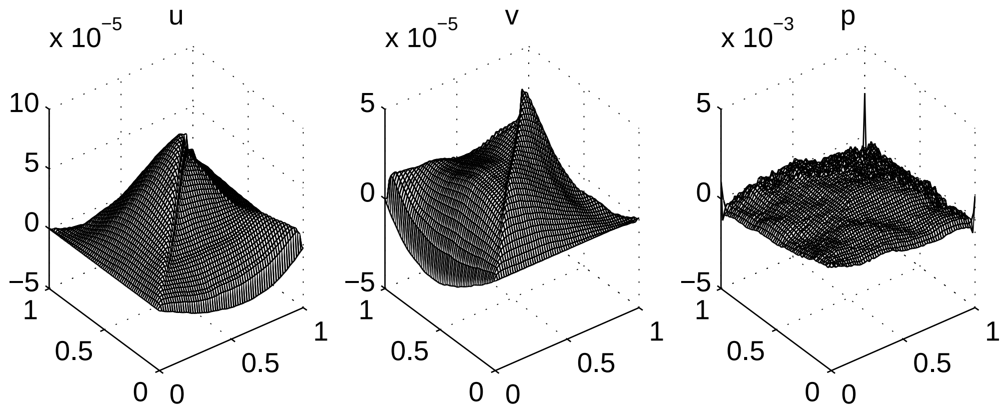


Figure 3.20: Error Functions for Algorithms 3.1 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

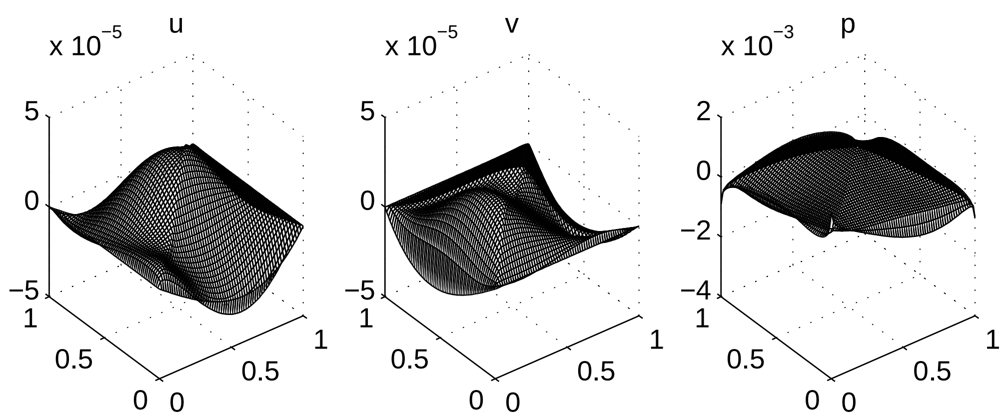


Figure 3.21: Error Functions for Algorithms 3.2 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

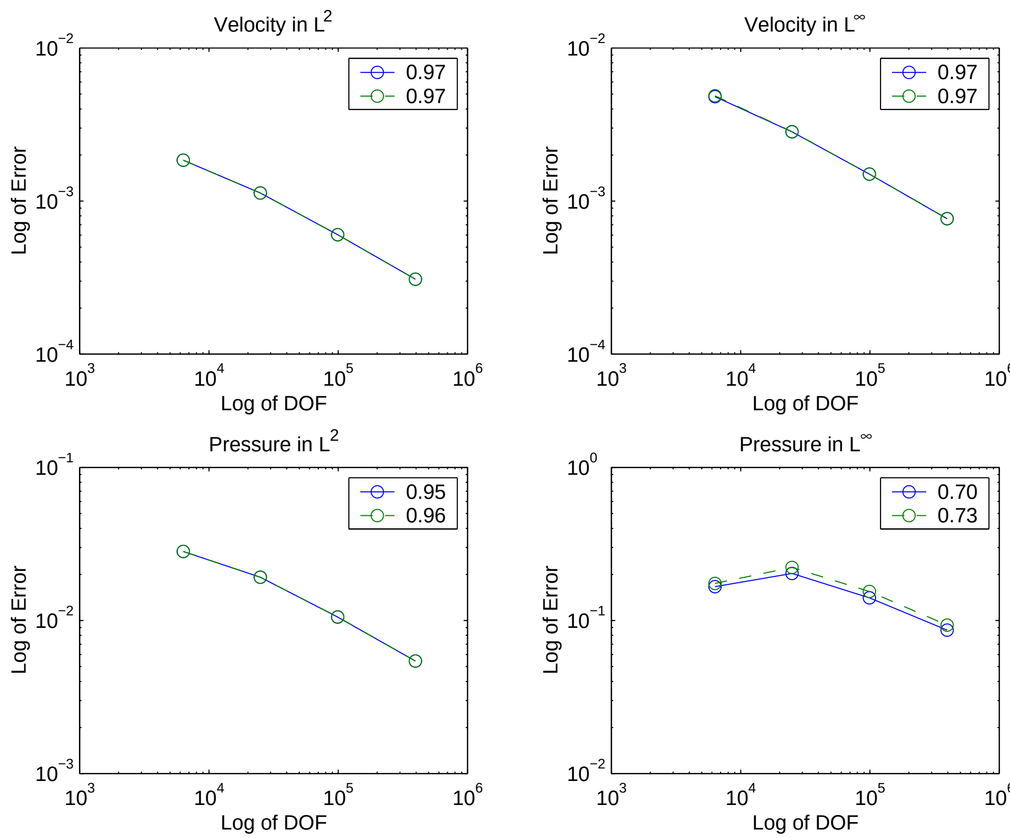


Figure 3.22: Error Decay of Algorithms 3.1 (Solid) and 3.2 (Dashed) with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

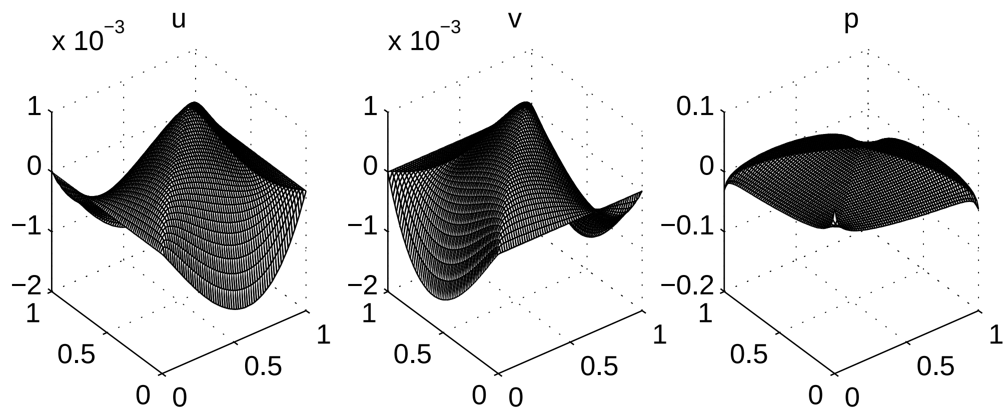


Figure 3.23: Error Functions for Algorithms 3.1 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

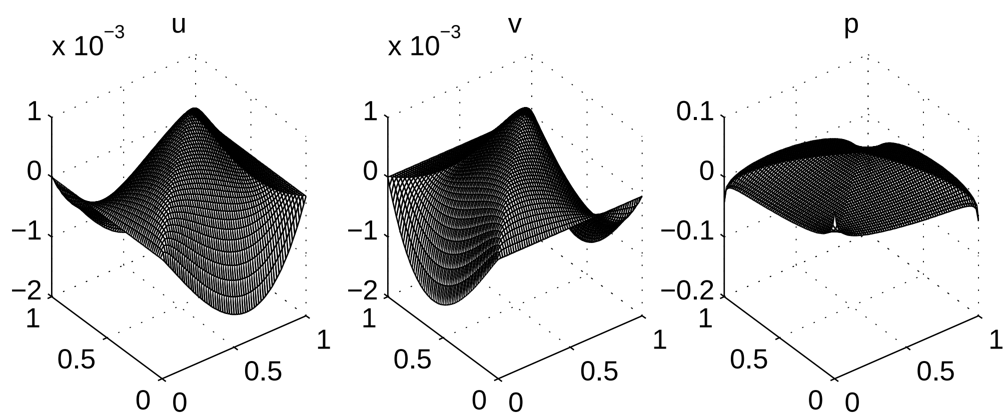


Figure 3.24: Error Functions for Algorithms 3.2 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

3.7.2 Algorithm 3.1 : $P_1 - P_1 - P_1$ on Regular Domain

The main crucial discovery for gauge method in these numerical experiments is that they do not depend on inf-sup condition for regular mesh which is symmetric and equidistance. The Figure 3.25 show us the errors both velocity and pressure for Algorithm 3.1 converge to 0 on the regular domain (b) in Figure 1.2 in contrast with the error for pressure on distorted mesh Figure 3.1 and 3.4. The pressures also for the Chorin-Uzawa and the Gauge-Uzawa methods do not converge to exact solution in $\mathbf{L}^\infty$ space (see Figure 2.13 and 5.9).

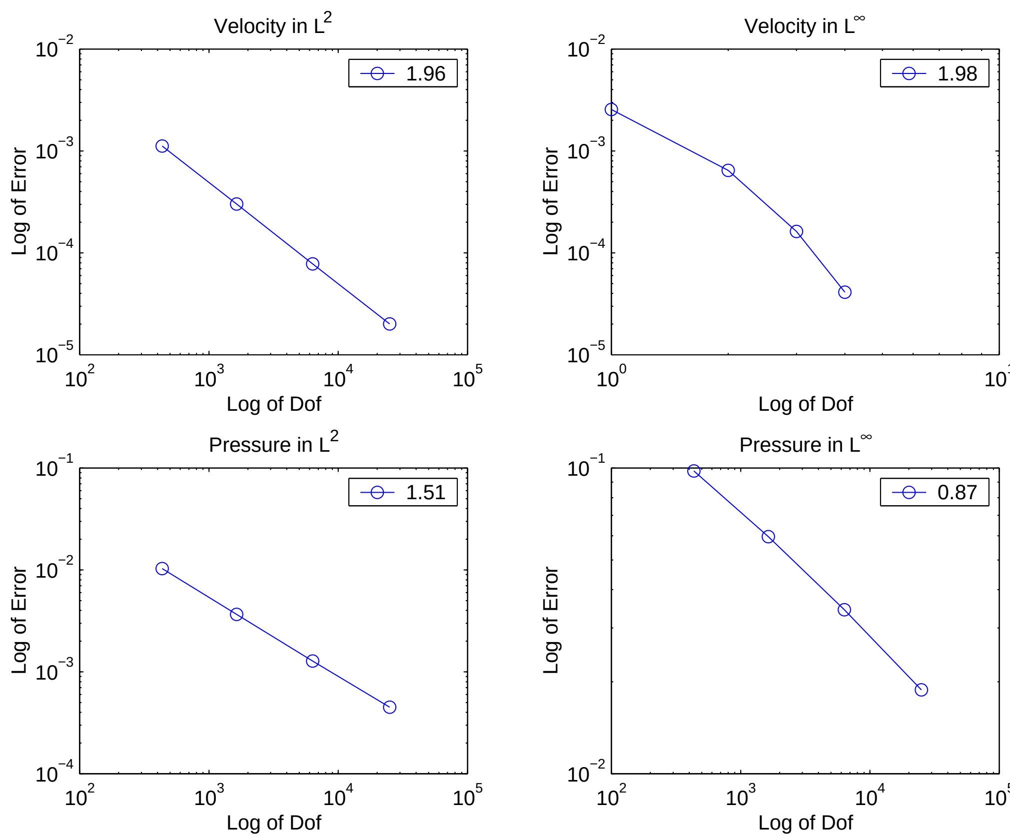


Figure 3.25: Error Decay of Algorithm 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2.

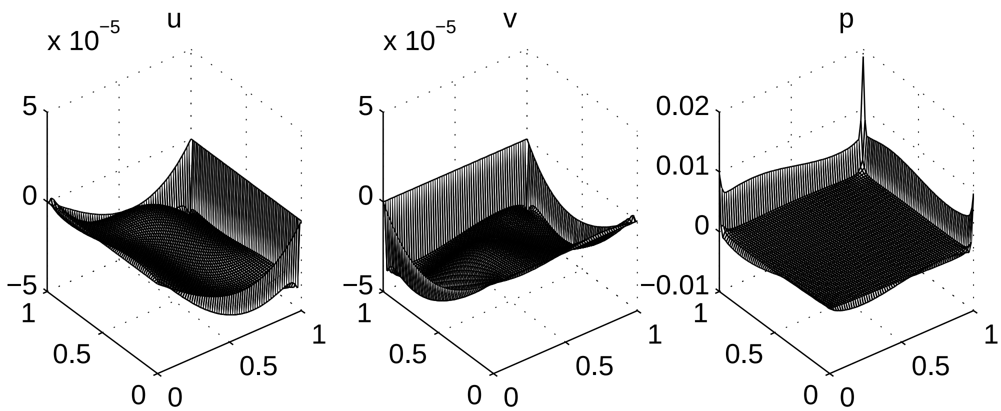


Figure 3.26: Error Functions of Algorithm 3.1 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963).

3.7.3 Algorithms 3.3 and 3.4 : Dirichlet Boundary Condition

As we mentioned in Remark 3.3, the pressures of Algorithms 3.3 and 3.4 do not necessary to converge to exact solution of NSE (1.1.1). Indeed, pressures in all experiments in this section are not stable.

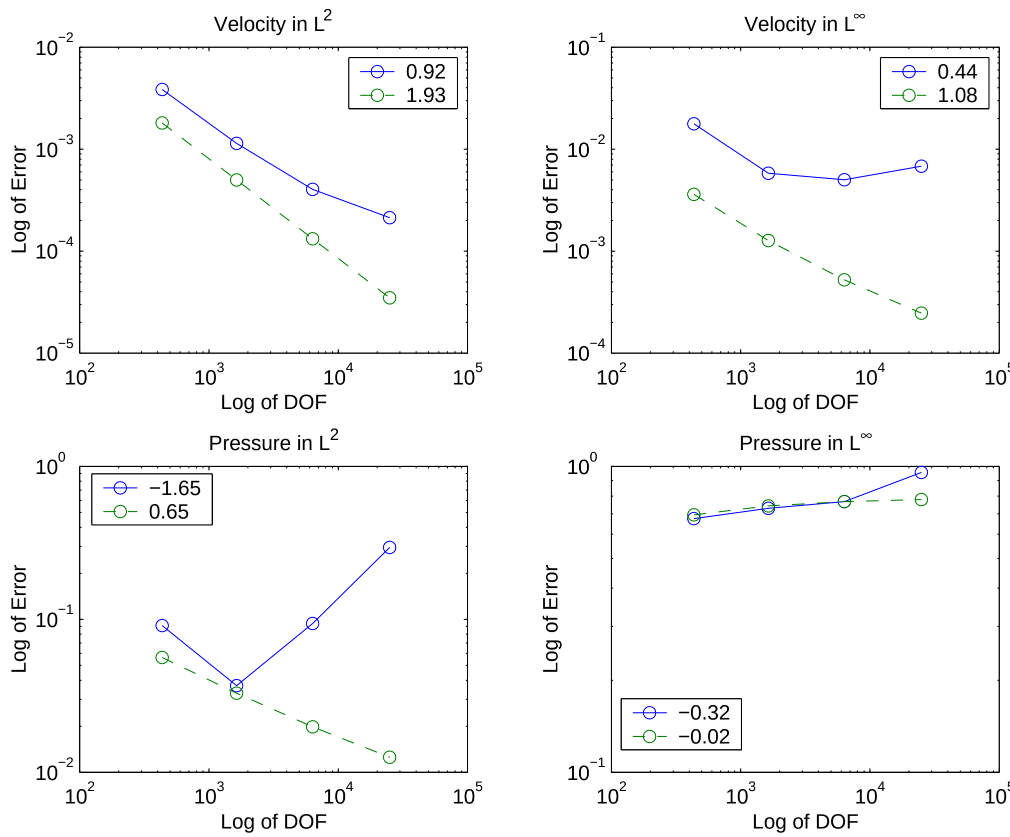


Figure 3.27: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements.

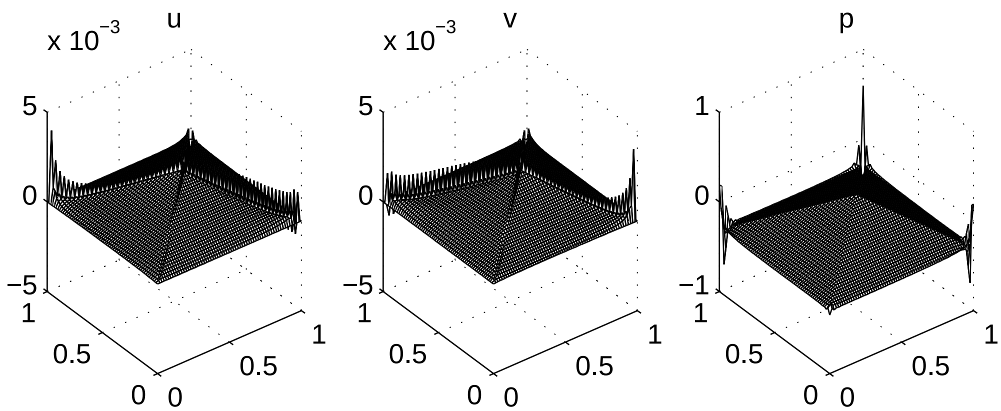


Figure 3.28: Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ (DOF = 24,963) Elements.

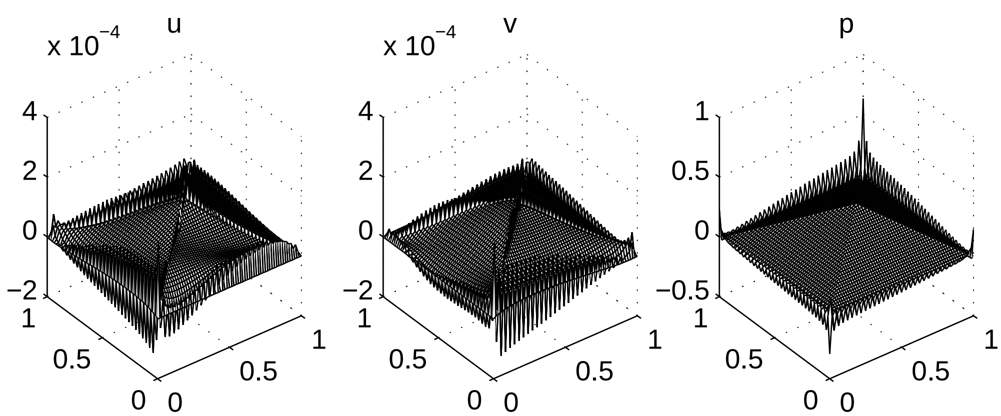


Figure 3.29: Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

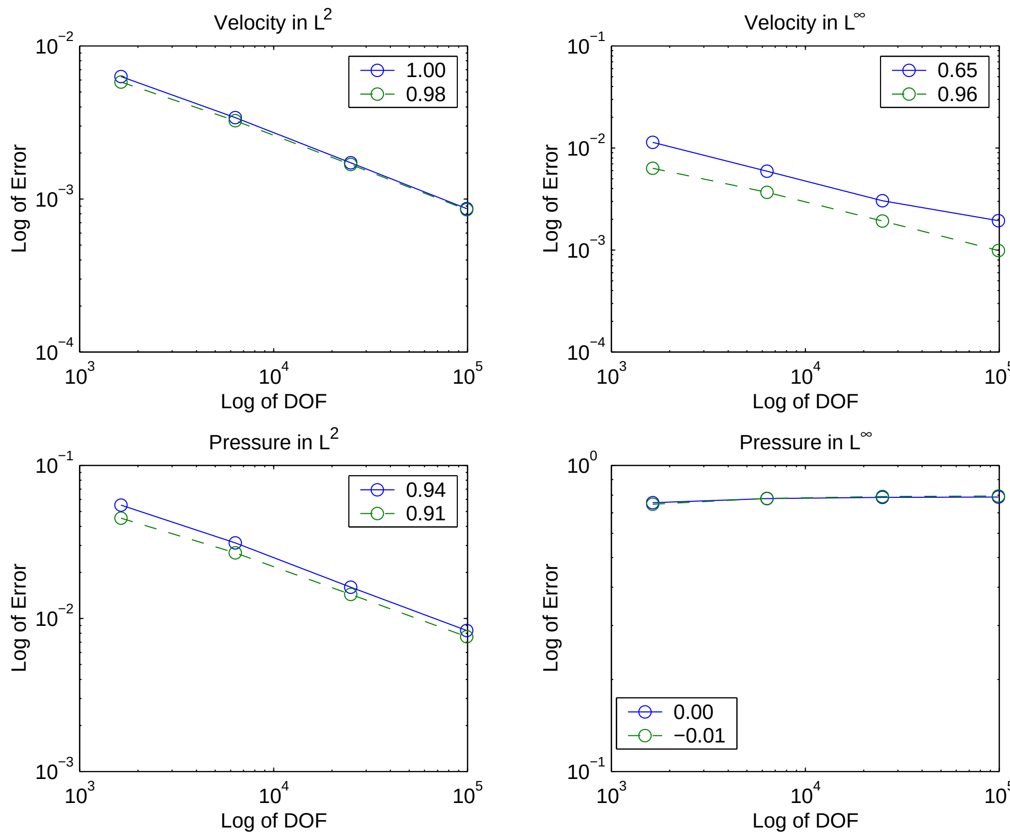


Figure 3.30: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements.

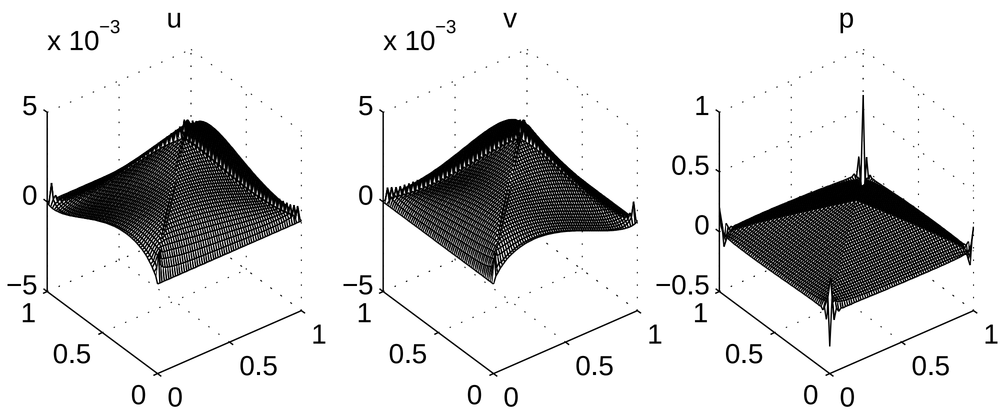


Figure 3.31: Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

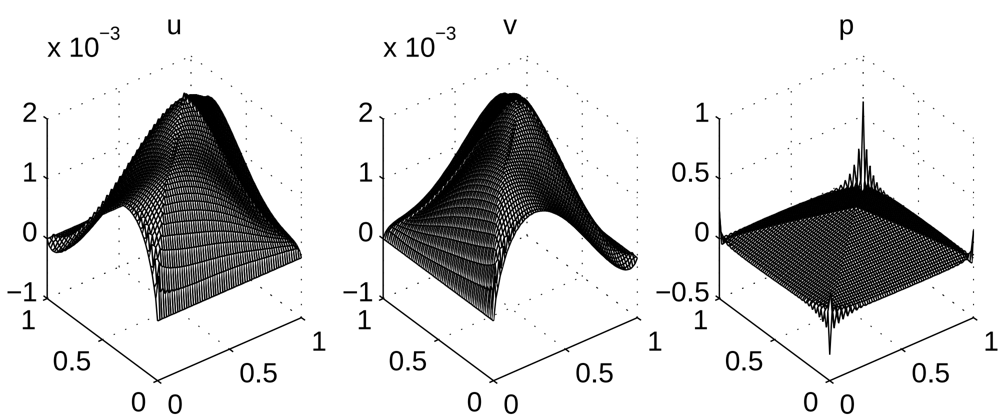


Figure 3.32: Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_1$ Elements (DOF = 24,963).

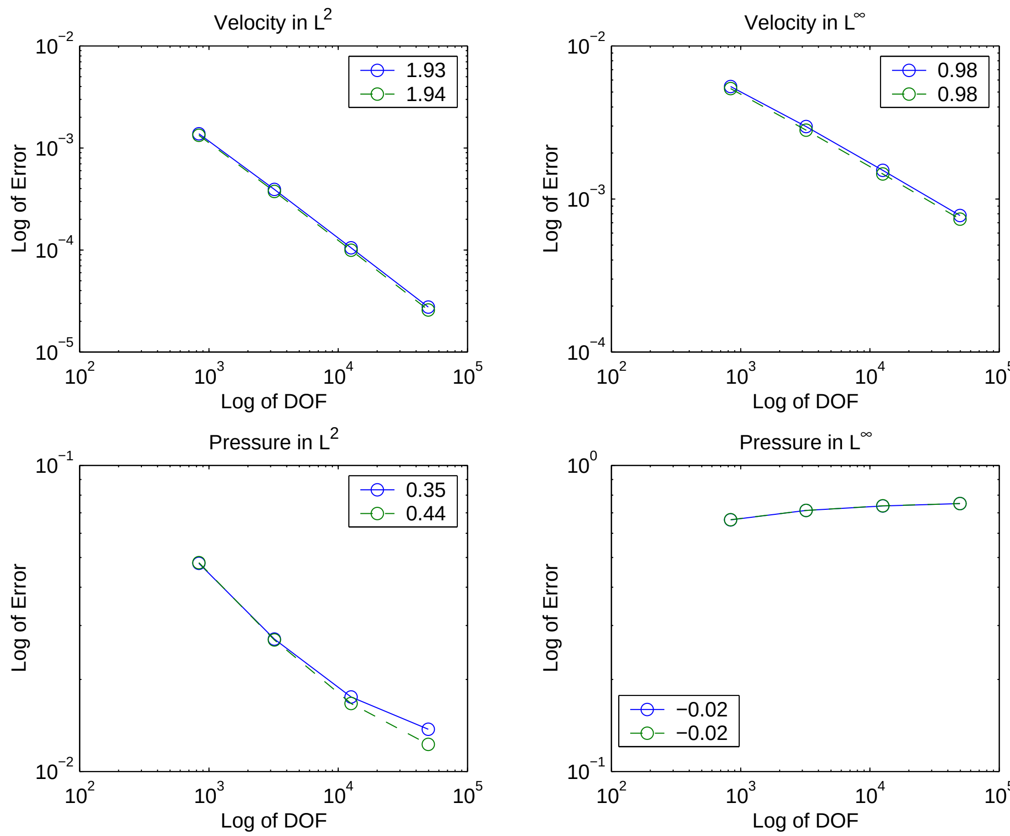


Figure 3.33: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements.

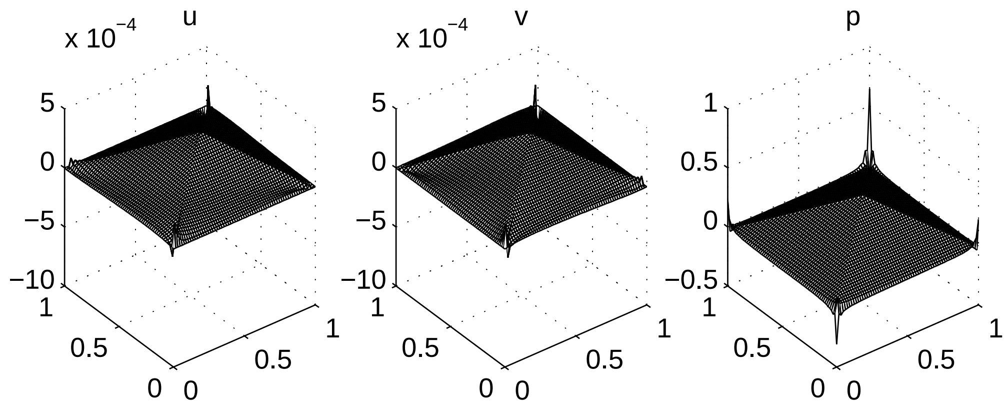


Figure 3.34: Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

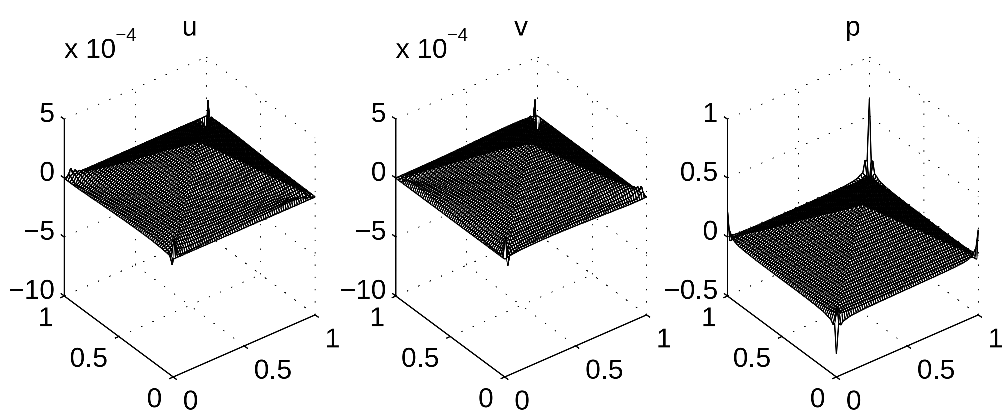


Figure 3.35: Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

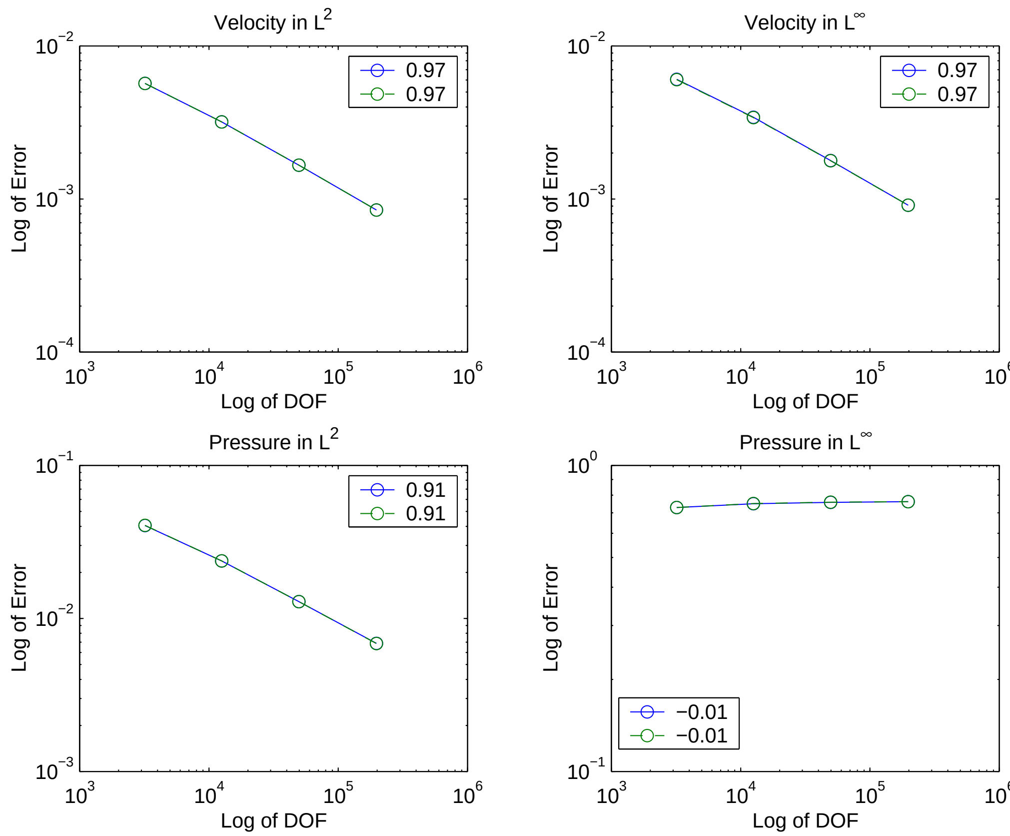


Figure 3.36: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements.

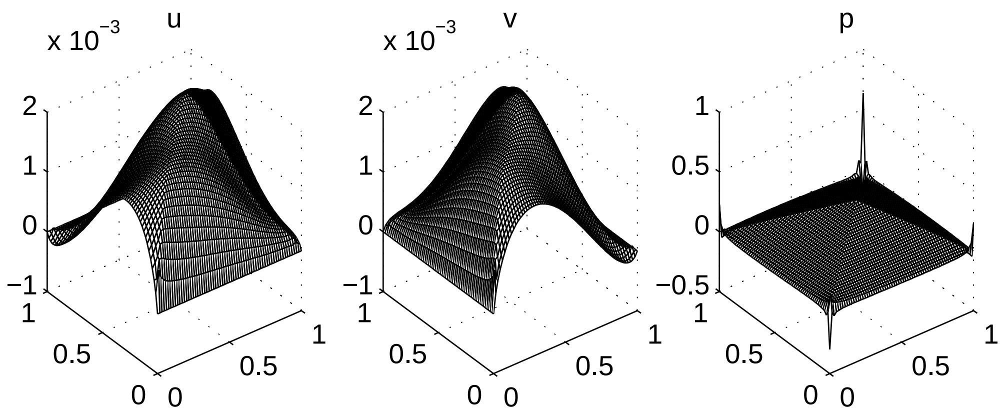


Figure 3.37: Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

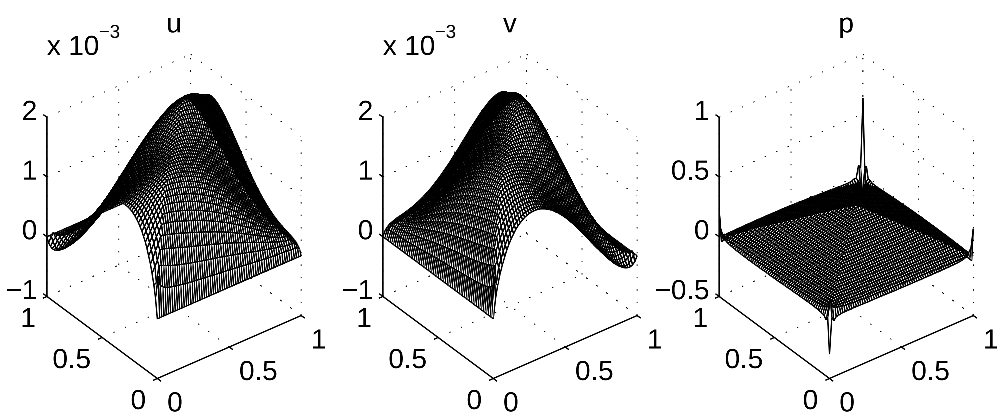


Figure 3.38: Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_1 - P_1 - P_2$ Elements (DOF = 49,667).

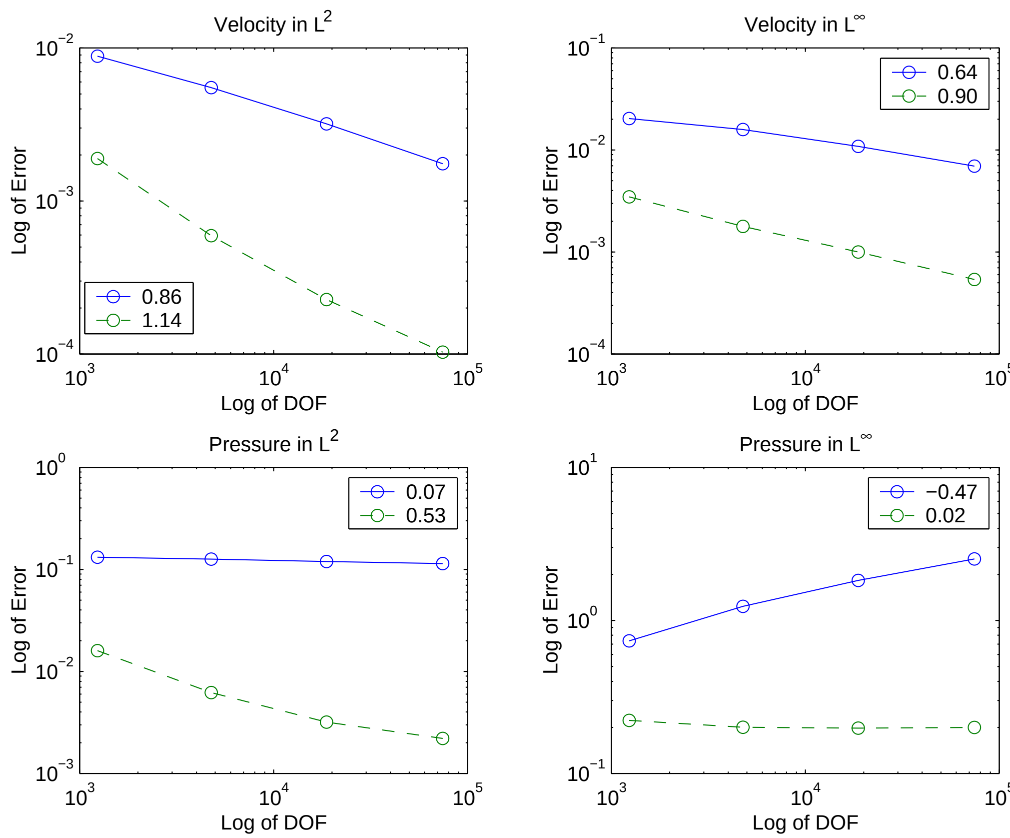


Figure 3.39: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements.

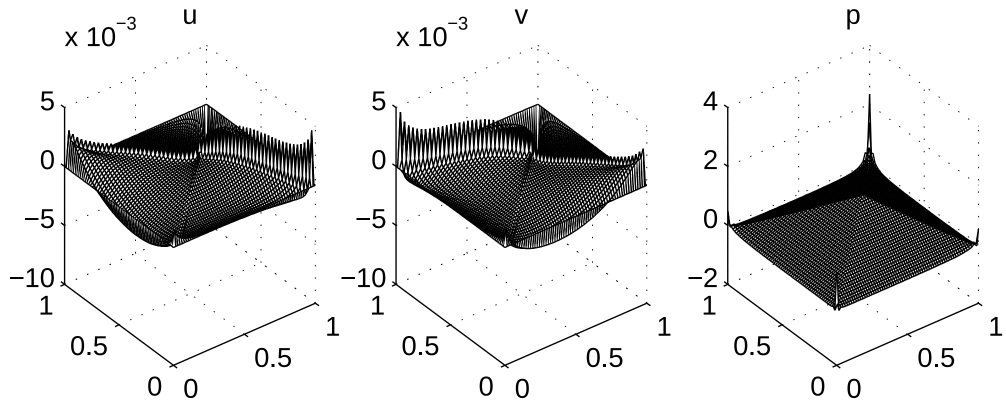


Figure 3.40: Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

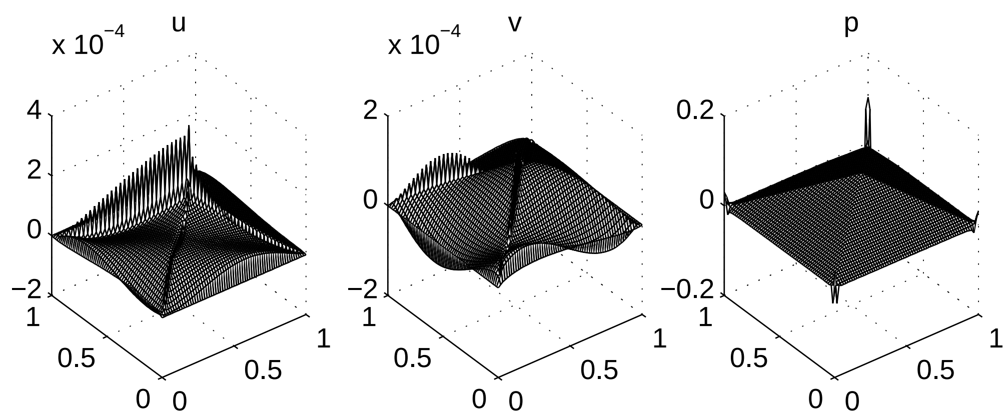


Figure 3.41: Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

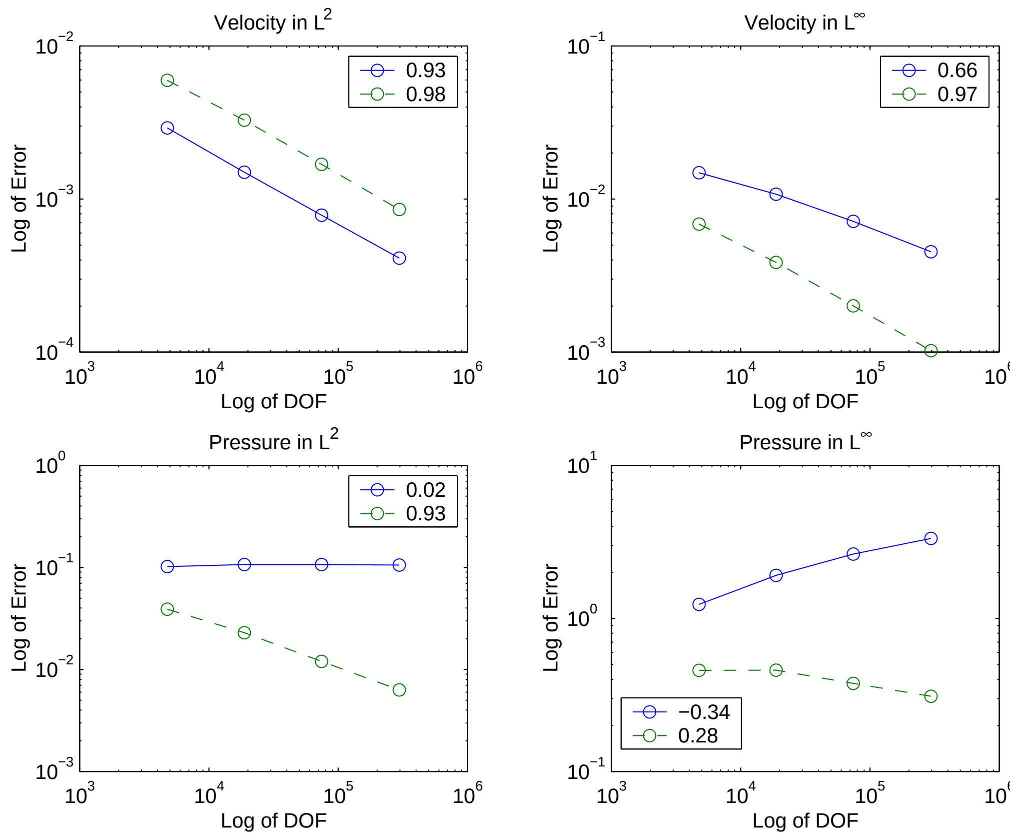


Figure 3.42: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_1$ Elements.

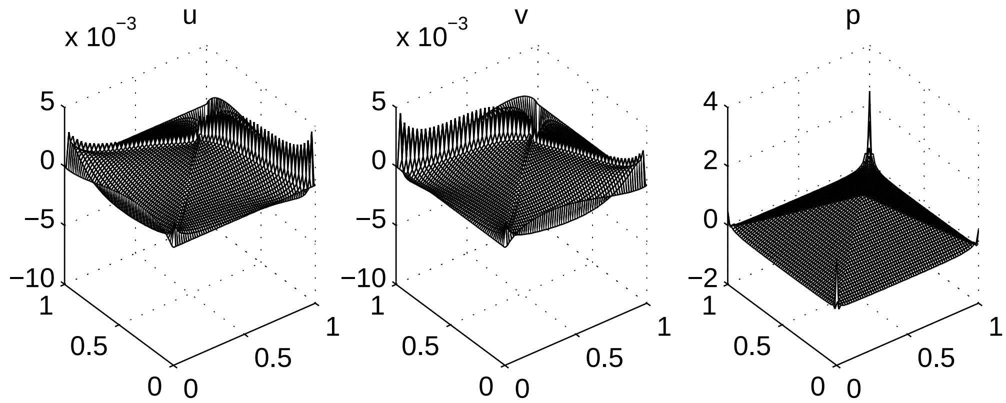


Figure 3.43: Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

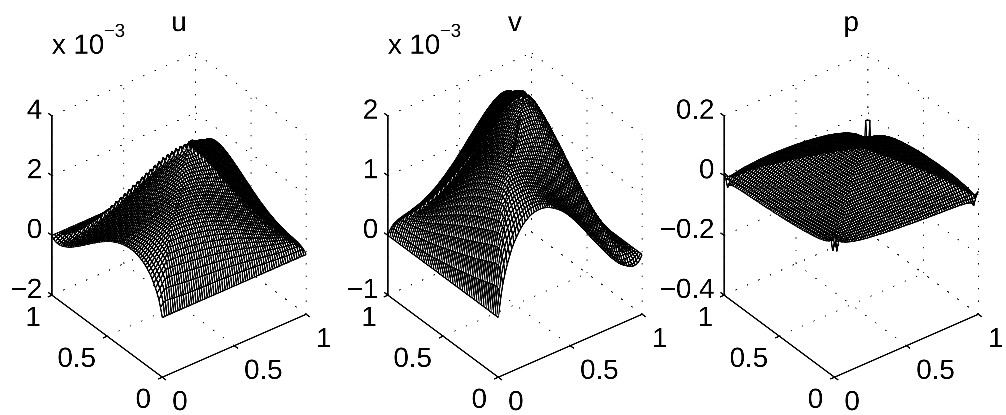


Figure 3.44: Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_1$ Elements (DOF = 74,371).

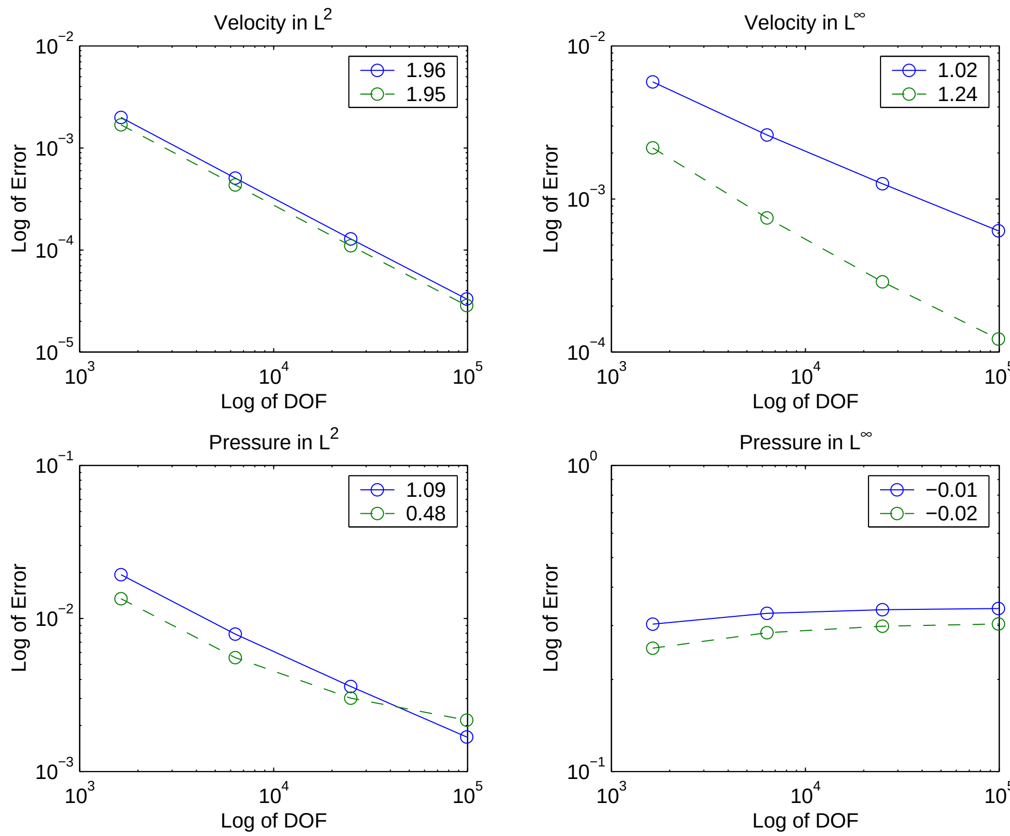


Figure 3.45: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements.

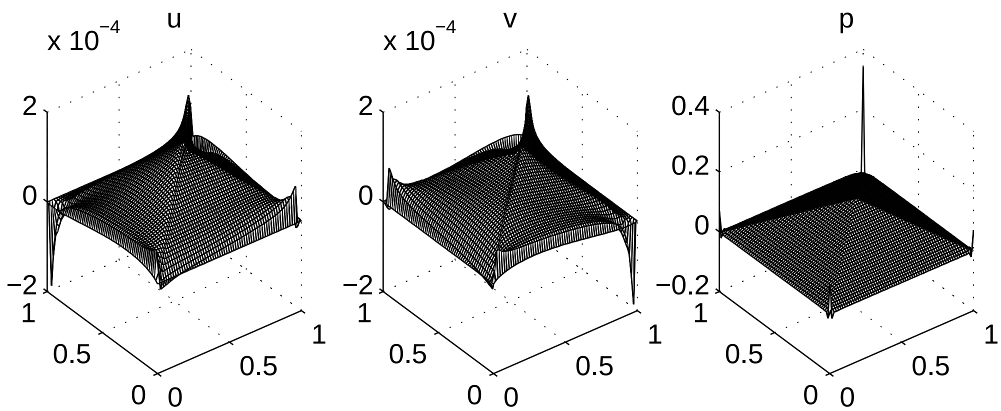


Figure 3.46: Error Functions for Algorithms 3.3 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

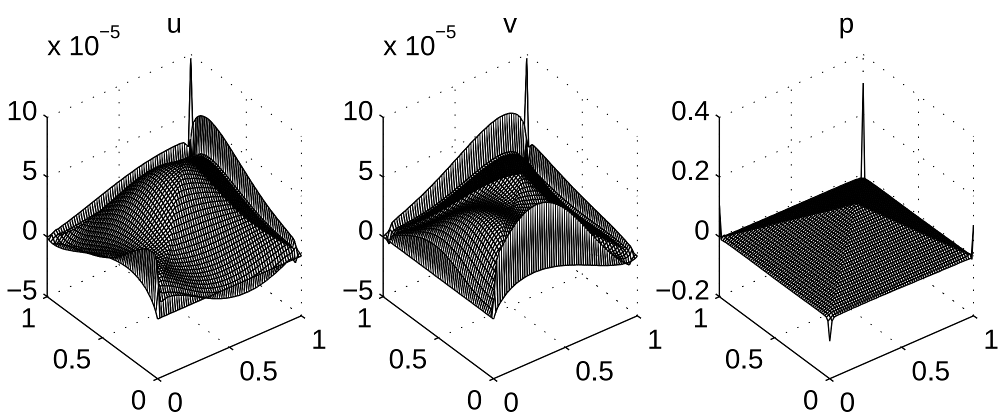


Figure 3.47: Error Functions for Algorithms 3.4 with $\triangle t = h^2$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

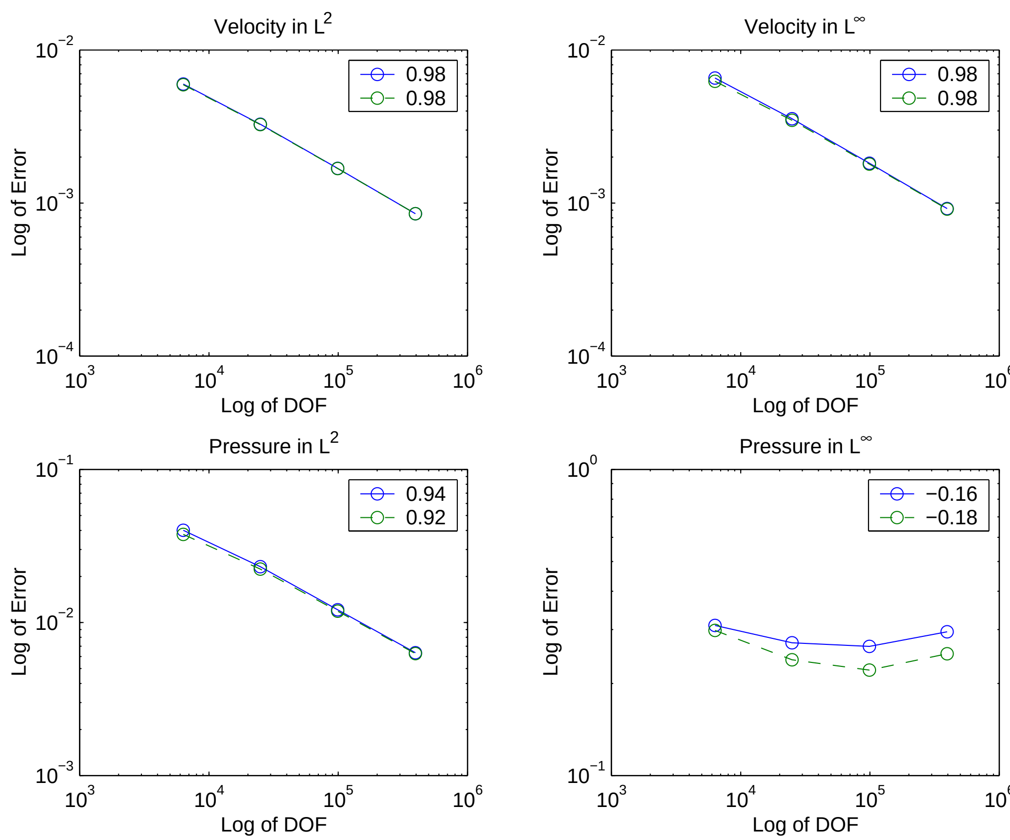


Figure 3.48: Error Decay of Algorithms 3.3 (Solid) and 3.4 (Dashed) with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

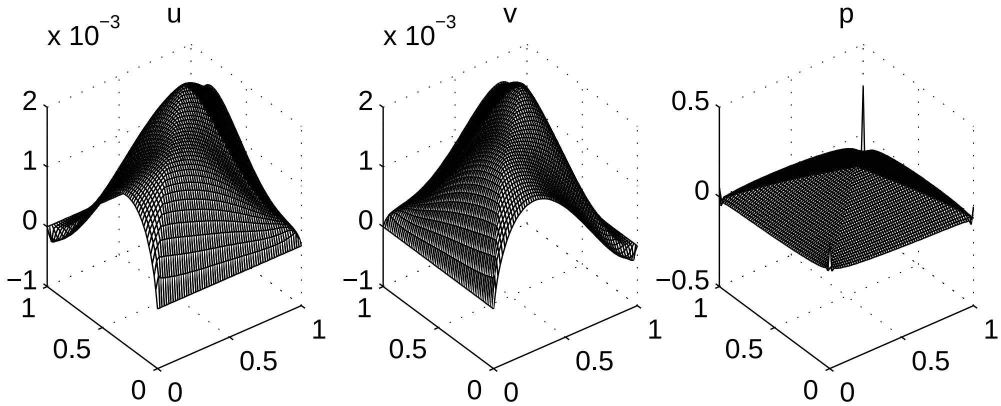


Figure 3.49: Error Functions for Algorithms 3.3 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

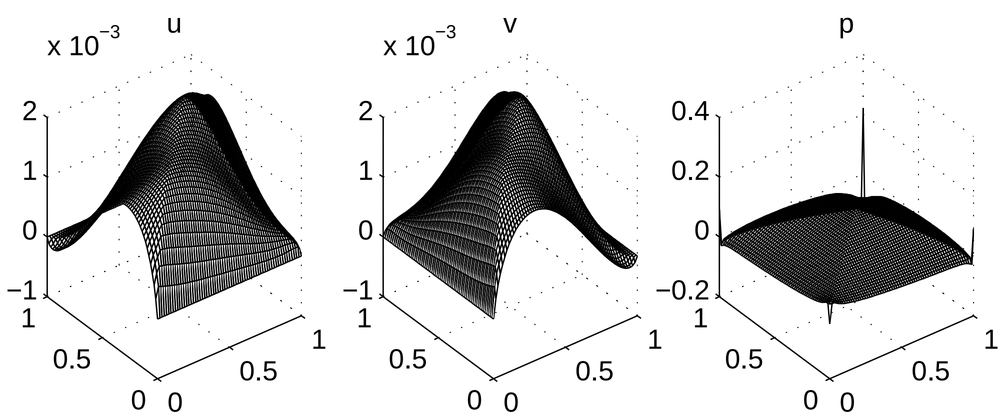


Figure 3.50: Error Functions for Algorithms 3.4 with $\triangle t = h$ and Spaces $P_2 - P_1 - P_2$ Elements (DOF = 99,075).

3.7.4 Example : Singular Solution

We obtain quite reasonable numerical results of gauge methods for the smooth example in subsection (3.7.1), but their performances for the singular solution are too bad to be applied to an unknown regularity problem. The corner singularity of the exact solution make big error on the boundary derivative, and it is represented as a pick on the corner even velocity in the Figure 3.52. This pick make the convergence of velocity error slow down, and the error of pressure increase in Figure 3.51

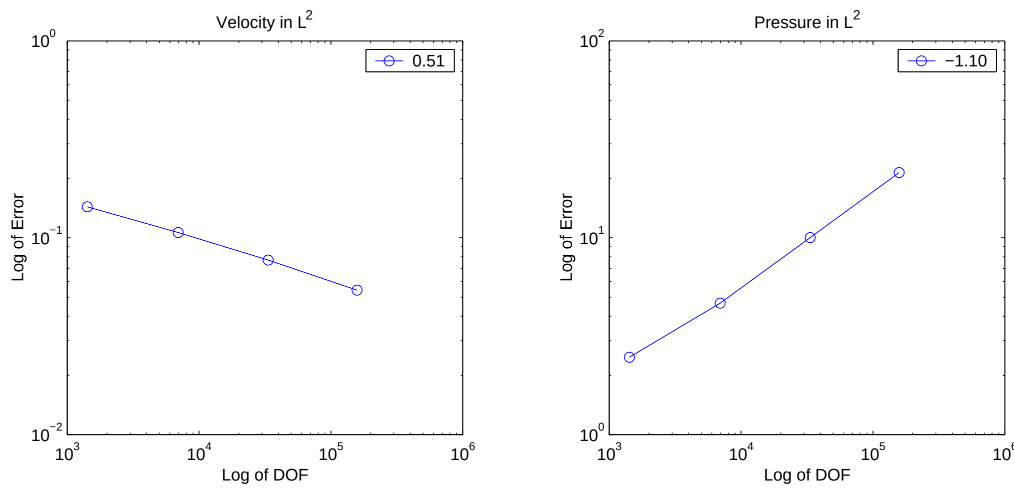
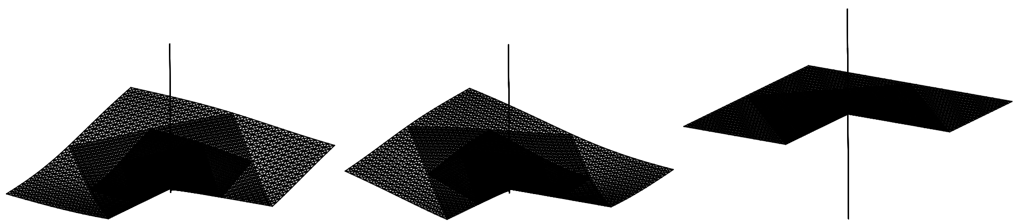


Figure 3.51: Error Decay of Gauge Method Algorithm 3.1 with $\triangle t = h$ and $P_2 - P_1 - P_3$ Elements.



u v p

Figure 3.52: Numerical Solution of Gauge Method Algorithm 3.1 with $\triangle t = h$ and $P_2 - P_1 - P_3$ Elements (DOF = 158,119).

Chapter 4

Iterative Solvers for the Stationary Stokes Equations

In this chapter we investigate the coupling of the finite element method with the the gauge method and related projection methods. Of special interest is the role of compatibility between the various discrete spaces involved (discrete inf-sup). In order to focus on space discretization alone, we study the stationary Stokes system:

$$\begin{cases} -\dfrac{1}{Re}\triangle \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega p \, d\Omega = 0. & \end{cases} \tag{4.0.1}$$

Since the gauge method requires the evaluation of boundary differentiation of phase variable ϕ, we discuss their variational computation in Section 4.1. We introduce a finite element gauge method in Section 4.2 and present several numerical experiments. They show that the discrete ϕ must have higher degree polynomial than that of velocity for stable computations. This represents a considerable computational cost associated with the gauge method, because ϕ is just an artificial variable. Reinterpreting the gauge method via change variables, we construct Gauge-Uzawa method in Section 4.4, and its rate of convergence. We also study the relation between Gauge-Uzawa and Uzawa methods. In Section 4.6, we find the rate of convergence for the Uzawa algorithm. Brezzi and Fortin [1] proved its convergence using the Schur complement properties, provided the relaxation parameter α for the Uzawa method satisfies $0 < \alpha < 1$. Our result is that the rate of convergence is $1 - 2\alpha\beta^2 + \alpha^2\beta^2$, where β is the inf-sup constant. On the other hand, This insists that the convergence range for α is from 0 to 2, and the optimal value for α is 1.

4.1 Variational computation of boundary differentiations

A key difficulty in actual computations with gauge methods is to provide accurate approximation of boundary derivatives $\dfrac{\partial \phi^{n+1}}{\partial \nu}$ or $\dfrac{\partial \phi^{n+1}}{\partial \tau}$ on $\partial\Omega$. We recall now a variational approximation of boundary derivatives. First we consider the Laplace equation

$$\begin{cases} -\triangle \phi = f, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

and approximation of the normal derivative $\dfrac{\partial \phi}{\partial \nu}$. Integrating (4.1.1) by parts against $\psi \in H^1(\Omega)$, we find the variational expression

$$-\int_\Omega \triangle \phi \psi d\mathbf{x} = -\int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} \psi d\Gamma + \int_\Omega \nabla \phi \nabla \psi d\mathbf{x} \tag{4.1.2}$$

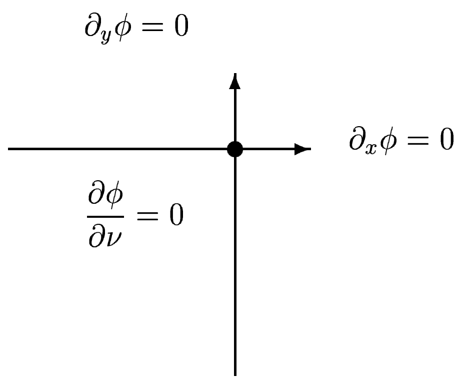


Figure 4.1: $\dfrac{\partial \phi}{\partial \nu} = 0$ at Each Corner Provided $\phi = 0$.

or

$$\int_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \psi d\Gamma = -\int_{\Omega} f \psi d\mathbf{x} + \int_{\Omega} \nabla \phi \nabla \psi d\mathbf{x}, \tag{4.1.3}$$

where the unit normal ν is well defined except at corners. Equality (4.1.3) defines $\dfrac{\partial \phi}{\partial \nu} \in H^{-\frac{1}{2}}(\partial \Omega)$ uniquely as a linear functional in $H^{\frac{1}{2}}(\partial \Omega)$ (Trace space of $H^1(\Omega)$). One goal is to use a similar expression to defined the discrete counterpart. To this end, we follow Pehlivanov et al [3, 20]. The first issue is the concept of *normal derivative*, at a corner. Since $\phi = 0$ on $\partial \Omega$, the tangential derivatives vanish, and so does $\nabla \phi$, at a corner (see Figure 4.1). We thus impose

$$\frac{\partial \phi}{\partial \nu} = 0 \quad \text{at corner of } \partial \Omega. \tag{4.1.4}$$

Let $\mathfrak{T} = K$ be a shape-regular quasi-uniform partition of Ω. Let $\mathbb{B}_h$ be a conforming finite element space containing piecewise linear and let $\mathbb{B}_h^b$ be the *boundary* finite element space

$$\mathbb{B}_h^b = \{w_h \in \mathbb{B}_h : w_h = 0 \text{ at the interior and corner nodes of } \Omega\}. \tag{4.1.5}$$

We also define

$$\mathbb{B}_h^0 = \{w_h \in H_0^1(\Omega) : w_h = 0 \text{ on } \partial \Omega\}. \tag{4.1.6}$$

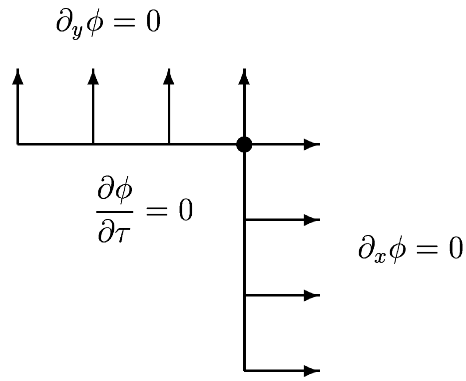


Figure 4.2: $\frac{\partial\phi}{\partial\tau} = 0$ at Each Corner Provided $\frac{\partial\phi}{\partial\nu} = 0$.

Let $\phi_h \in \mathbb{B}_h^0$ be the finite element solution of (4.1.1), namely,

$$\phi_h \in \mathbb{B}_h^0 : \int_\Omega \nabla\phi_h \nabla\psi_h dx = \int_\Omega f\psi_h dx, \quad \forall\psi_h \in \mathbb{B}_h^0. \tag{4.1.7}$$

In view of (4.1.3), we define the approximate normal derivative $\partial_\nu\phi_h$ to be:

$$\phi_h \in \mathbb{B}_h^0 : \int_{\partial\Omega} \partial_\nu\phi_h\psi_h d\Gamma = -\int_\Omega f\psi_h d\mathbf{x} + \int_\Omega \nabla\phi_h\nabla\psi_h d\mathbf{x}, \quad \forall\phi_h \in \mathbb{B}_h^b. \tag{4.1.8}$$

Lemma 4.1 *If $f \in H^2(\Omega)$ and $\phi \in H^3(\Omega)$, then*

$$\|\partial_\nu\phi - \partial_\nu\phi_h\|_{0,\Gamma} \le Ch^{\frac{3}{2}}(\|\phi\|_{3,\Omega} + \|f\|_{2,\Omega}). \tag{4.1.9}$$

In Algorithms 3.3-3.4 of Chapter 3, as well as Algorithm 4.1 and 4.2 below, derivative $\partial_\nu\phi_h^n$ can be calculated by the variational formula:

$$\partial_\nu\phi_h^n \in \mathbb{B}_h^b : \int_{\partial\Omega} \partial_\nu\phi_h^n\psi_h d\Gamma = -\int_\Omega \text{div } \mathbf{a}_h^n\psi_h d\mathbf{x} + \int_\Omega \nabla\phi_h^n\nabla\psi_h d\mathbf{x}, \quad \forall\psi_h \in \mathbb{B}_h^b. \tag{4.1.10}$$

Now we consider the approximation of tangential derivative $\partial_\tau\phi$ on $\partial\Omega$ provided ϕ does no longer vanish on $\partial\Omega$. Integration by parts of $-\triangle\phi = f$ yields for all

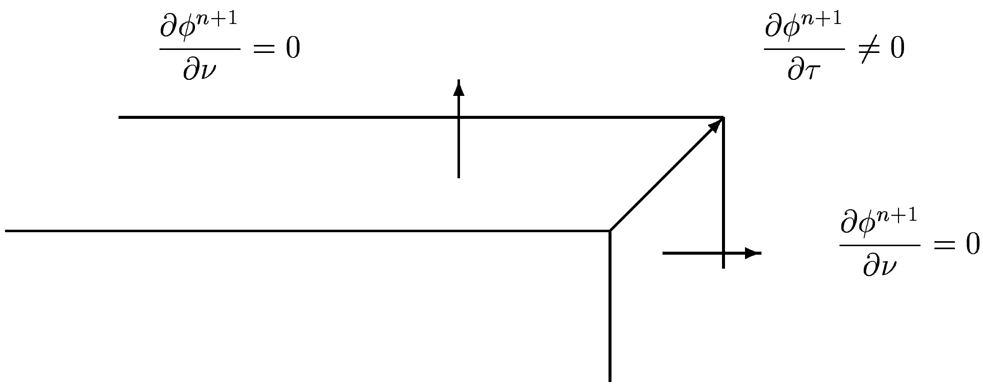


Figure 4.3: Difficulty of Variational Formula in 3D

$\psi \in H^1(\Omega)$

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial \tau} \psi d\Gamma = \int_{\Omega} \nabla \phi \,\mathbf{curl}\, \psi d\mathbf{x}. \tag{4.1.11}$$

If $\phi^n \in \mathbb{B}_h$ is the finite element approximation of ϕ^n, then the discrete of (4.1.11) reads:

$$\partial_\tau \phi_h^n \in \mathbb{B}_h^b : \quad \int_{\partial\Omega} \frac{\partial \phi_h^n}{\partial \tau} \psi_h d\Gamma = \int_{\Omega} \nabla \phi_h^n \,\mathbf{curl}\, \psi_h d\mathbf{x}, \quad \forall \psi_h \in H^1(\Omega). \tag{4.1.12}$$

Formula (4.1.12) can be used to approximate $\partial_\nu \phi_h^n$ in 2d. However, in 3d we have two orthogonal tangential differentiations, and (4.1.12) do not make sense any longer. Moreover, the tangential derivative along an edge in 3d may not vanish even though $\partial_\nu \phi = 0$ (see Figure 4.3). These are two serious limitations for the use of Algorithms 3.1-3.2 in 3d, which are those better behaved in 2d. Dealing with boundary derivatives is indeed a critical computational issue, and a serious drawback of gauge methods. We will show in Section 4.4 how to reformulate the gauge method, thereby giving rise to the Gauge-Uzawa method which preserves all advantages of the the former but does not deal with boundary derivatives.

4.2 Space Discretization via Gauge Method

The purpose of this section is to introduce a finite element gauge method for the stationary Stokes problem and discuss compatibility conditions among the discrete space for velocity $\mathbf{u}$, pressure p, and gauge variable ϕ. To introduce a finite element discretization, we let $(\mathbb{V}_h, \mathbb{P}_h)$ be the discrete spaces for velocity and pressure and $\mathbb{R}_h$ be

$$\begin{aligned} &\mathbb{R}_h = \{\gamma_h \in \mathbf{C}^0(\Omega) : \gamma_h|_K \in \mathcal{P}(K), \text{ for all } K \in \mathfrak{T}\} \quad \text{and} \\ &\mathbb{R}_h^0 = \{\gamma_h \in \mathbb{R}_h : \gamma_h = 0 \text{ on } \partial\Omega\} \end{aligned} \tag{4.2.1}$$

where $\mathcal{P}(K)$ is a polynomial spaces of degree ≥ 1 fixed independent of $K \in \mathfrak{T}$. To compute finite element solution $\mathbf{a}_h^{n+1}$ which is not 0 on $\partial\Omega$, we define a extension operators χ_ν and χ_τ from $\mathbb{B}_h^b$ to $\mathbb{V}_h$ such that $\forall g_h \in \mathbb{B}_h^b$,

$$\begin{cases} \chi_\nu(g_h) = 0, & \chi_\tau(g_h) = 0, & \text{at all interior nodes,} \\ \chi_\nu(g_h)\cdot\nu = -g_h, & \chi_\tau(g_h)\cdot\nu = 0, & \text{at boundary nodes,} \\ \chi_\tau(g_h)\cdot\tau = 0, & \chi_\tau(g_h)\cdot\tau = -g_h, & \text{at boundary nodes.} \end{cases} \tag{4.2.2}$$

We estimate easily $\chi_\nu(g_h) = (u_h, v_h)$ by solving the linear system

$$\begin{cases} u_h\cdot\nu_1 + u_h\cdot\nu_2 = -g_h \\ v_h\cdot\nu_1 + v_h\cdot\nu_2 = 0, \end{cases} \tag{4.2.3}$$

and vanishing at all interior nodes. The value $\chi_\tau(g_h)$ can be done similarly. Then $\mathbf{a}_h^{n+1}$ can be split by $\mathbf{a}_h^{n+1} = \bar{\mathbf{a}}_h^{n+1} + \chi_\tau(\partial_\tau\phi_h^n)$, where $\bar{\mathbf{a}}_h^{n+1}$ is a function in $\mathbb{V}_h^0$. The gauge Algorithm 3.1 in the present simpler setting can be regarded as an iterative procedure.

Algorithm 4.1 (Gauge Algorithm 3.1 for the Stationary Stokes problem) *Start with initial value* $\phi_h^0 = 0$.

Step 1: *Find* $\bar{\mathbf{a}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\frac{1}{Re}\left\langle \nabla \bar{\mathbf{a}}_h^{n+1}, \nabla \mathbf{v}_h \right\rangle = \langle \mathbf{f}, \mathbf{v}_h \rangle - \frac{1}{Re}\left\langle \nabla \chi_\tau(\partial_\tau \phi_h^n), \nabla \mathbf{v}_h \right\rangle, \qquad \forall \mathbf{v}_h \in \mathbb{V}_h^0, \tag{4.2.4}$$

Step 2: *Find* $\mathbf{a}_h^{n+1} \in \mathbb{V}_h$

$$\mathbf{a}_h^{n+1} = \bar{\mathbf{a}}_h^{n+1} + \chi_\tau(\partial_\tau \phi_h^n), \tag{4.2.5}$$

Step 3: *Find* $\phi_h^{n+1} \in \mathbb{R}_h$ *such that*

$$\left\langle \nabla \phi_h^{n+1}, \nabla \delta_h \right\rangle = \left\langle \operatorname{div} \mathbf{a}_h^{n+1}, \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{R}_h, \tag{4.2.6}$$

Step 4: *Find* $\partial_\tau \phi_h^{n+1} \in \mathbb{B}_h^b$ *such that*

$$\left\langle \partial_\tau \phi_h^{n+1}, \delta_h \right\rangle_\Gamma = -\left\langle \operatorname{div} \mathbf{a}_h^{n+1}, \delta_h \right\rangle + \left\langle \mathbf{curl}\, \phi_h^{n+1}, \mathbf{curl}\, \delta_h \right\rangle \qquad \forall \delta_h \in \mathbb{B}_h^b, \tag{4.2.7}$$

Step 5: *Find* $\bar{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\left\langle \bar{\mathbf{u}}_h^{n+1}, \mathbf{v}_h \right\rangle = \left\langle \mathbf{a}_h^{n+1} + \chi_\tau(\partial_\tau \phi_h^{n+1} - \partial_\tau \phi_h^n), \mathbf{v}_h \right\rangle + \left\langle \nabla \phi_h^{n+1}, \mathbf{v}_h \right\rangle, \qquad \forall \mathbf{v}_h \in \mathbb{V}_h^0, \tag{4.2.8}$$

Step 6: *Find* $\mathbf{u}_h^{n+1}$ *such as*

$$\mathbf{u}_h^{n+1} = \bar{\mathbf{u}}_h^{n+1} - \chi_\tau(\partial_\tau \phi_h^{n+1} - \partial_\tau \phi_h^n). \tag{4.2.9}$$

Step 7: *Find* $p_h^{n+1} \in \mathbb{P}_h$ *such as*

$$\left\langle p_h^{n+1}, \delta_h \right\rangle = -\frac{1}{Re}\left\langle \nabla \phi_h^{n+1}, \nabla \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{P}_h. \tag{4.2.10}$$

Remark 4.2 We need to do iteration steps 1-4 for Algorithm 4.1, because the values of Steps 5-7 is not required to estimate for the first 4 steps.

Similarly, the gauge Algorithm 3.2 can be viewed as an iterative procedure to approximate (4.0.1).

Algorithm 4.2 *(*Gauge Algorithm 3.2 for the Stationary Stokes Problem*) Start with initial value* $\phi^0 = 0$.

Step 1: *Find* $\bar{\mathbf{a}}_h^{n+1} \in \mathbb{V}_h^0$ *such that (4.2.4),*

Step 2: *Find* $\mathbf{a}_h^{n+1} \in \mathbb{V}_h$ *such that (4.2.5),*

Step 3: *Find* $\psi_h^{n+1} \in \mathbb{R}_h^0$ *such that*

$$\left\langle \nabla \psi_h^{n+1} \,,\, \nabla \delta_h \right\rangle = \left\langle \mathrm{rot}\, \mathbf{a}^{n+1} \,,\, \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{R}_h^0, \tag{4.2.11}$$

Step 4: *Find* $\partial_\nu \phi_h^{n+1} \in \mathbb{B}_h^b$ *such that*

$$\langle \partial_\nu \psi_h \,,\, \delta_h \rangle_\Gamma = -\left\langle \mathrm{rot}\, \mathbf{a}_h^{n+1} \,,\, \delta_h \right\rangle + \left\langle \nabla \phi_h^{n+1} \,,\, \nabla \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{B}_h^b. \tag{4.2.12}$$

Step 5: *Find* $\bar{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\left\langle \bar{\mathbf{u}}_h^{n+1} \,,\, \mathbf{v}_h \right\rangle = \left\langle \mathbf{curl}\, \psi^{n+1} \,,\, \mathbf{v}_h \right\rangle + \langle \chi_\tau(\partial_\nu \psi_h) \,,\, \mathbf{v}_h \rangle \,, \forall \mathbf{v}_h \in \mathbb{V}_h^0 \tag{4.2.13}$$

Step 6: *Find* $\mathbf{u}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\mathbf{u}_h^{n+1} = \bar{\mathbf{u}}_h^{n+1} - \chi_\tau(\partial_\nu \psi_h), \tag{4.2.14}$$

Step 7: *Find* $\phi_h^{n+1} \in \mathbb{V}_h$ *such that*

$$\nabla \phi^{n+1} = \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, \tag{4.2.15}$$

Step 8: *Find* $\chi_\nu(\partial_\nu \phi^{n+1})$ *from* $\nabla \phi^{n+1}$ *in step 7,*

Step 9: *Find* p_h^{n+1} *such that*

$$\left\langle p^{n+1}\,,\,\delta_h\right\rangle = -\frac{1}{Re}\left\langle \text{div}\;\mathbf{a}_h^{n+1}\,,\,\delta_h\right\rangle \tag{4.2.16}$$

Remark 4.3 We need to do iteration steps 1-8 for Algorithm 4.2, because the pressure in Step 9 is not required to estimate in the iteration iteration.

The gauge Algorithm 3.3 can be can be represented for (4.0.1).

Algorithm 4.3 *(*Gauge Algorithm 3.3 for the Stationary Stokes problem*) Start with initial value* $\phi_h^0 = 0$.

Step 1: *Find* $\bar{\mathbf{a}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\frac{1}{Re}\left\langle \nabla\bar{\mathbf{a}}_h^{n+1}\,,\,\nabla\mathbf{v}_h\right\rangle = \langle \mathbf{f}\,,\,\mathbf{v}_h\rangle - \frac{1}{Re}\left\langle \nabla\chi_\nu(\partial_\nu\phi_h^n)\,,\,\nabla\mathbf{v}_h\right\rangle, \quad \forall\mathbf{v}_h \in \mathbb{V}_h^0, \tag{4.2.17}$$

Step 2: *Find* $\mathbf{a}_h^{n+1} \in \mathbb{V}_h$

$$\mathbf{a}_h^{n+1} = \bar{\mathbf{a}}_h^{n+1} + \chi_\nu(\partial_\nu\phi_h^n), \tag{4.2.18}$$

Step 3: *Find* $\phi_h^{n+1} \in \mathbb{R}_h^0$ *such that*

$$\left\langle \nabla\phi_h^{n+1}\,,\,\nabla\delta_h\right\rangle = \left\langle \text{div}\;\mathbf{a}_h^{n+1}\,,\,\delta_h\right\rangle, \quad \forall\delta_h \in \mathbb{R}_h^0, \tag{4.2.19}$$

Step 4: *Find* $\partial_\tau\phi_h^{n+1} \in \mathbb{B}_h^b$ *such that*

$$\left\langle \partial_\nu\phi_h^{n+1}\,,\,\delta_h\right\rangle_\Gamma = -\left\langle \text{div}\;\mathbf{a}_h^{n+1}\,,\,\delta_h\right\rangle + \left\langle \nabla\phi_h^{n+1}\,,\,\nabla\delta_h\right\rangle, \quad \forall\delta_h \in \mathbb{B}_h^b, \tag{4.2.20}$$

Step 5: *Find* $\bar{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\left\langle \bar{\mathbf{u}}_h^{n+1}\,,\,\mathbf{v}_h\right\rangle = \left\langle \mathbf{a}_h^{n+1} + \chi_\nu(\partial_\nu\phi_h^{n+1} - \partial_\nu\phi_h^n)\,,\,\mathbf{v}_h\right\rangle + \left\langle \nabla\phi_h^{n+1}\,,\,\mathbf{v}_h\right\rangle, \quad \forall\mathbf{v}_h \in \mathbb{V}_h^0, \tag{4.2.21}$$

Step 6: *Find* $\mathbf{u}_h^{n+1}$ *such as*

$$\mathbf{u}_h^{n+1} = \bar{\mathbf{u}}_h^{n+1} - \chi_\nu(\partial_\nu \phi_h^{n+1} - \partial_\nu \phi_h^n). \tag{4.2.22}$$

Step 7: *Find* $p_h^{n+1} \in \mathbb{P}_h$ *such as*

$$\left\langle p_h^{n+1}\,,\, \delta_h \right\rangle = -\frac{1}{Re}\left\langle \operatorname{div} \mathbf{a}_h^{n+1}\,,\, \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{P}_h. \tag{4.2.23}$$

Remark 4.4 Algorithm 4.3 is needed to iterate steps 1-4. As we already talked in Remark 3.3, the pressure for Algorithm (4.3) does not converge to exact solution.

The gauge Algorithm 3.4 can be can be represented for (4.0.1).

Algorithm 4.4 *(*Gauge Algorithm 3.4 for the Stationary Stokes Problem*) Start with initial value* $\phi^0 = 0$.

Step 1: *Find* $\bar{\mathbf{a}}_h^{n+1} \in \mathbb{V}_h^0$ *such that (4.2.17),*

Step 2: *Find* $\mathbf{a}_h^{n+1} \in \mathbb{V}_h$ *such that (4.2.18),*

Step 3: *Find* $\psi_h^{n+1} \in \mathbb{R}_h$ *such that*

$$\left\langle \nabla \psi_h^{n+1}\,,\, \nabla \delta_h \right\rangle = \left\langle \operatorname{rot} \mathbf{a}^{n+1}\,,\, \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{R}_h, \tag{4.2.24}$$

Step 4: *Find* $\partial_\tau \phi_h^{n+1} \in \mathbb{B}_h^b$ *such that*

$$\langle \partial_\tau \psi_h\,,\, \delta_h \rangle_\Gamma = -\left\langle \operatorname{rot} \mathbf{a}_h^{n+1}\,,\, \delta_h \right\rangle + \left\langle \mathbf{curl}\, \phi_h^{n+1}\,,\, \mathbf{curl}\, \delta_h \right\rangle, \quad \forall \delta_h \in \mathbb{B}_h^b. \tag{4.2.25}$$

Step 5: *Find* $\bar{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\left\langle \bar{\mathbf{u}}_h^{n+1}\,,\, \mathbf{v}_h \right\rangle = \left\langle \mathbf{curl}\, \psi^{n+1}\,,\, \mathbf{v}_h \right\rangle + \langle \chi_\nu(\partial_\tau \psi_h)\,,\, \mathbf{v}_h \rangle, \qquad \forall \mathbf{v}_h \in \mathbb{V}_h^0 \tag{4.2.26}$$

Step 6: *Find* $\mathbf{u}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\mathbf{u}_h^{n+1} = \bar{\mathbf{u}}_h^{n+1} - \chi_\nu(\partial_\tau \psi_h), \tag{4.2.27}$$

Step 7: *Find* $\phi_h^{n+1} \in \mathbb{V}_h$ *such that*

$$\nabla \phi^{n+1} = \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, \tag{4.2.28}$$

Step 8: *Find* $\chi_\nu(\partial_\nu \phi^{n+1})$ *from* $\nabla \phi^{n+1}$ *in step 7,*

Step 9: *Find* p_h^{n+1} *such that*

$$\left\langle p^{n+1} \, , \, \delta_h \right\rangle = -\frac{1}{Re} \left\langle \text{div } \mathbf{a}_h^{n+1} \, , \, \delta_h \right\rangle \tag{4.2.29}$$

Remark 4.5 Algorithm 4.4 is needed to iterate steps 1-8. As we already talked in Remark 3.3, the pressure for Algorithm (4.4) does not converge to exact solution.

4.3 Simulations and Conclusions for the Gauge method

We report here numerical experiments with Algorithms 4.1-4.4 and examples 1.3.1 for several finite element spaces for velocity, pressure, and gauge variable as follows:

$$P_1 - P_1 - P_1, \quad P_1 - P_1 - P_2, \tag{4.3.1}$$

and

$$P_2 - P_1 - P_1, \quad P_2 - P_1 - P_2, \quad P_2 - P_1 - P_3. \tag{4.3.2}$$

All finite elements are continuous, and the pair $P_2 - P_1$ is the well known stable Taylor-Hood combination. We stress however that the combination of (4.3.1) do not satisfy the discrete inf-sup, and that Algorithms 4.1-4.2 do not necessarily converge to the solution of the discrete saddle point formulation of (4.0.1). We also point out that the calculation (4.2.10) and (4.2.16) of pressure is numerically unstable (numerical differentiation). We present experiments for Dirichlet Algorithms 4.3-4.4 in subsection 4.3.2.

4.3.1 Numerical Experiments for Algorithms 4.1-4.2

We first discuss for Algorithm 4.1. We observe the oscillations of velocity and pressure error across the interelement boundaries of the macrotriangulation of Figure 1.1 (a) (Lake of cancellation effects), and pressure peaks at the corners. The latter are so pronounced that there is no pointwise convergence of pressure for spaces $P_1 - P_1 - P_1$ and $P_2 - P_1 - P_1$ of Figures 4.4-4.10. Convergence of pressure is restored by increasing the polynomial degree of ϕ (see Figures 4.7, 4.13, 4.16). The spaces $P_2 - P_1 - P_3$ of Figures 4.17 and 4.18 lead to the smallest error for both velocity and pressure. We next discuss for Algorithm (4.2). In striking contrast with Algorithm 4.1, the unstable spaces $P_1 - P_1 - P_1$ of Figures 4.4 and 4.6 show pointwise convergence and better accuracy in $\mathbf{L}^2$. All other experiments are similar to those of Algorithm 4.2.

In summary, we have the following conclusions for Algorithms 4.1-4.2:

- There seems to be a compatibility condition between spaces for pressure p and phase variable ϕ (the latter must be of higher degree than the former);
- The spaces of velocity and pressure seem to be less critically coupled (see space $P_1 - P_1 - P_2$);

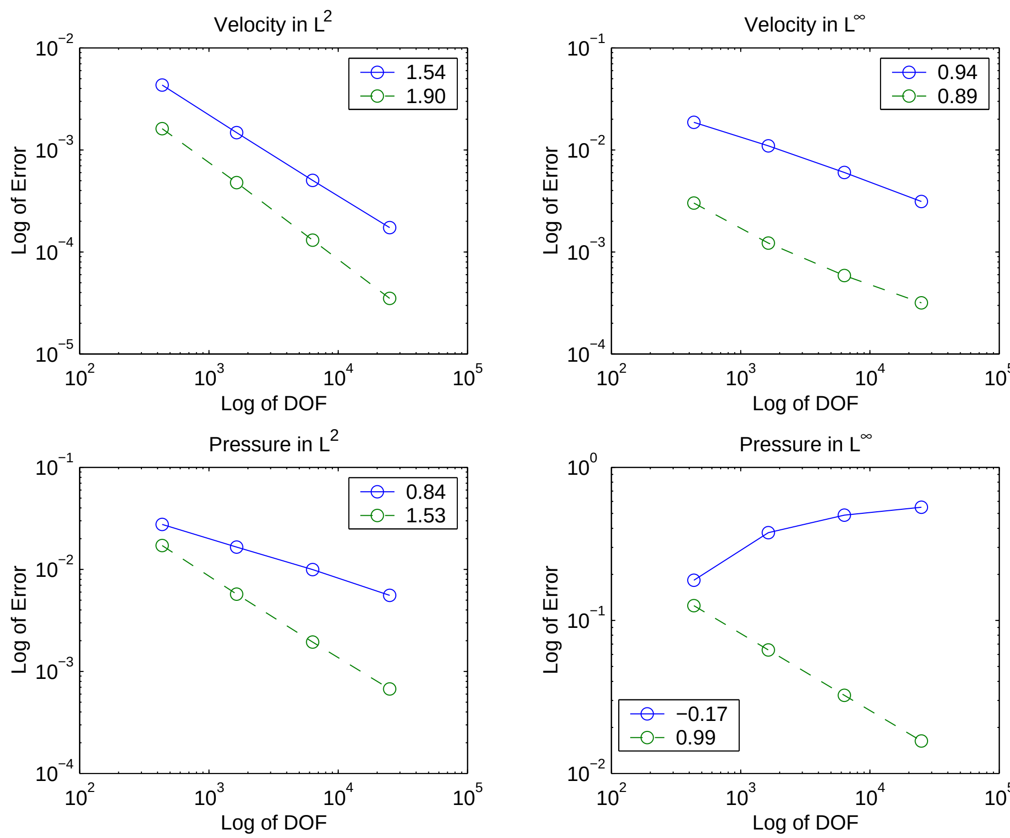


Figure 4.4: Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_1 - P_1 - P_1$.

- The best space combination is $P_2 - P_1 - P_3$ (see Figure 4.16), but there is a substantial computational effort in dealing with P_3. This combination performs similarly to the $P_2 - P_1$ Gauge-Uzawa method of Figure 4.34 below, which requires much less computational work and is thus preferable;

- The link between discrete space in via the boundary condition in (4.2.4)-(4.2.5). We do not know how to apply the saddle point theory of [1, 12] to this unusual formulation.

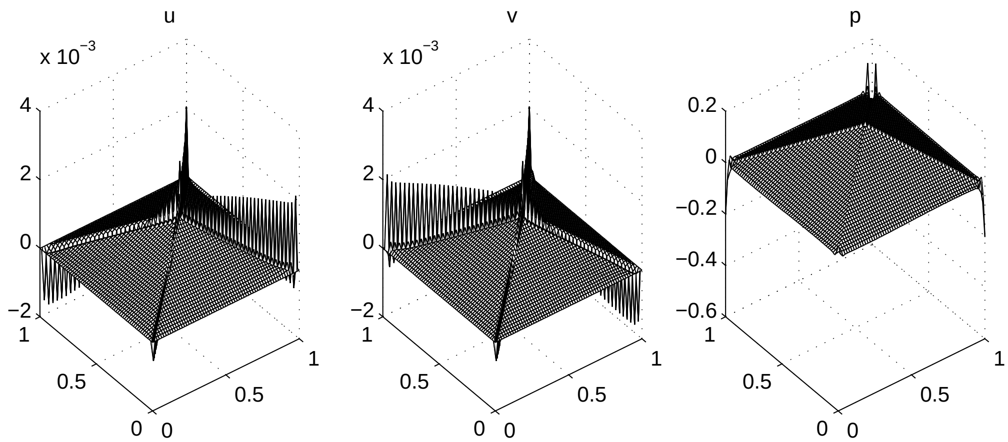


Figure 4.5: Error functions for Algorithm 4.1 with $P_1 - P_1 - P_1$.

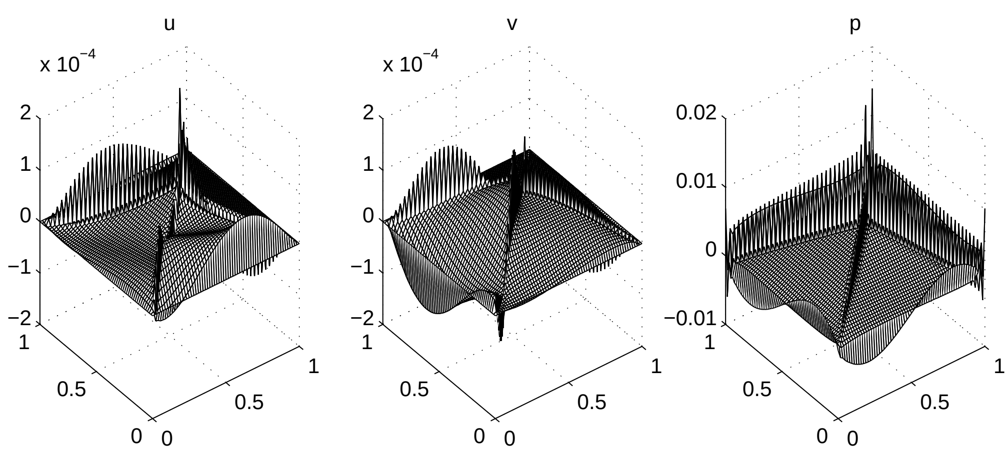


Figure 4.6: Error functions for Algorithm 4.2 with $P_1 - P_1 - P_1$.

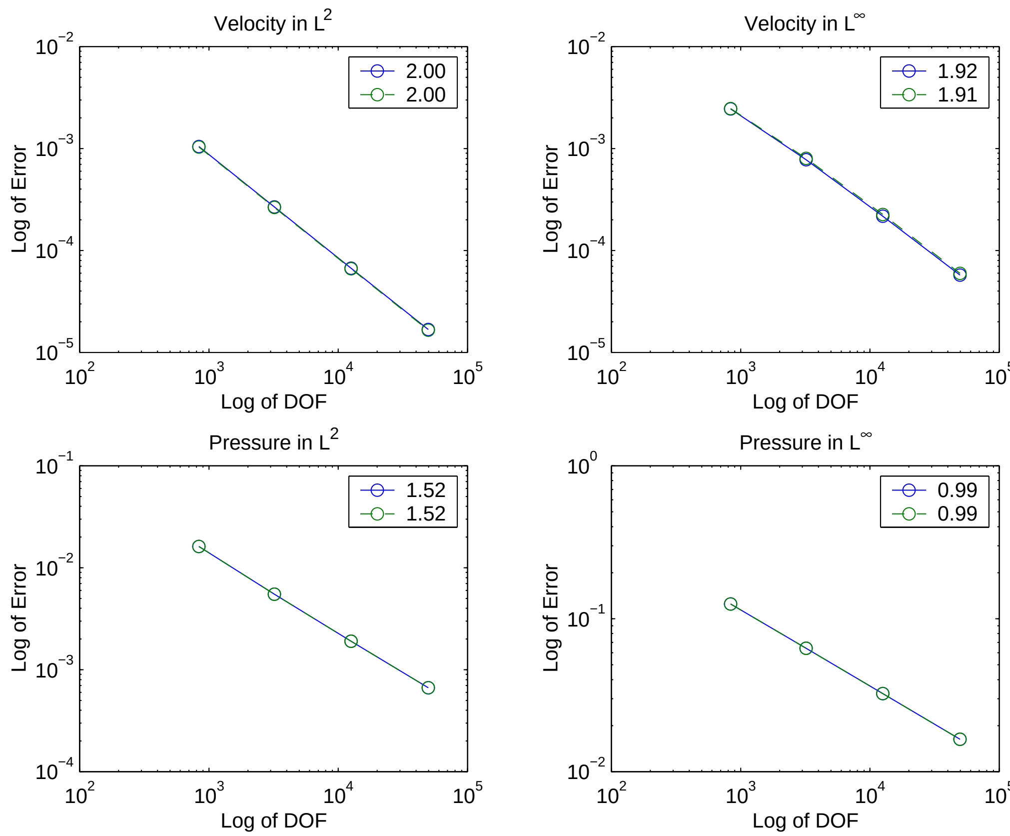


Figure 4.7: Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_1 - P_1 - P_2$

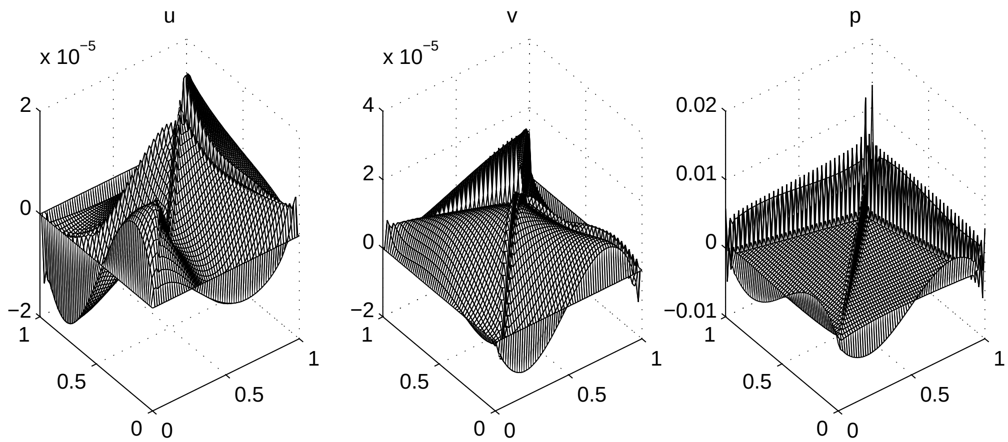


Figure 4.8: Error functions for Algorithm 4.1 with $P_1 - P_1 - P_2$.

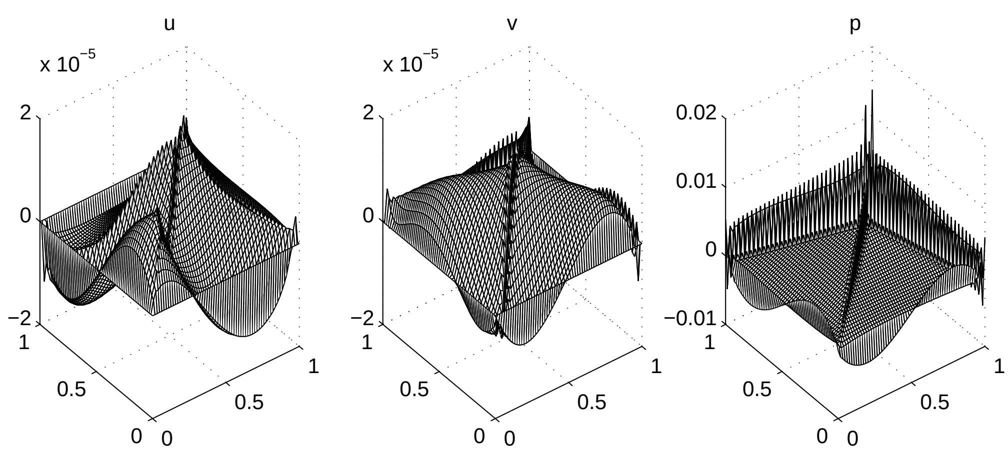


Figure 4.9: Error functions for Algorithm 4.2 with $P_1 - P_1 - P_2$.

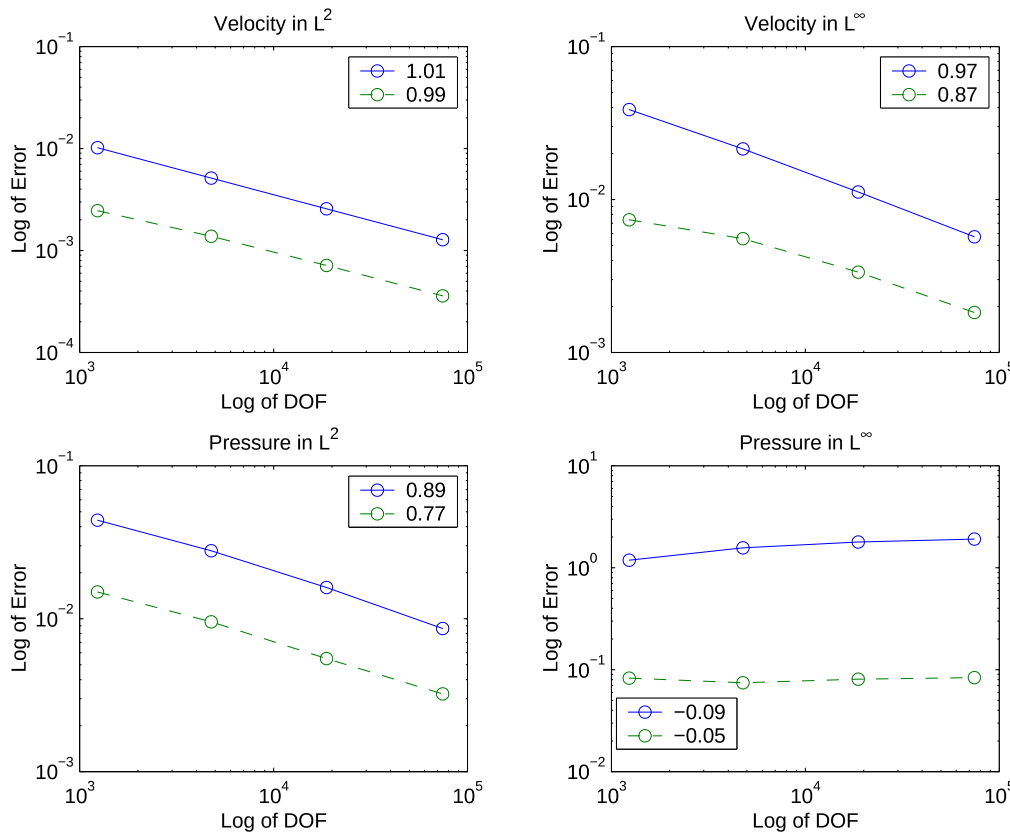


Figure 4.10: Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_1$

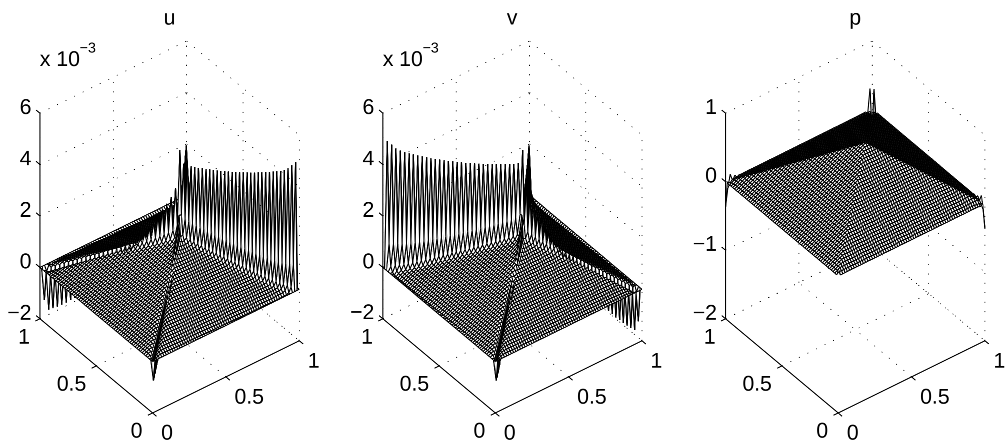


Figure 4.11: Error functions for Algorithm 4.1 with $P_2 - P_1 - P_1$.

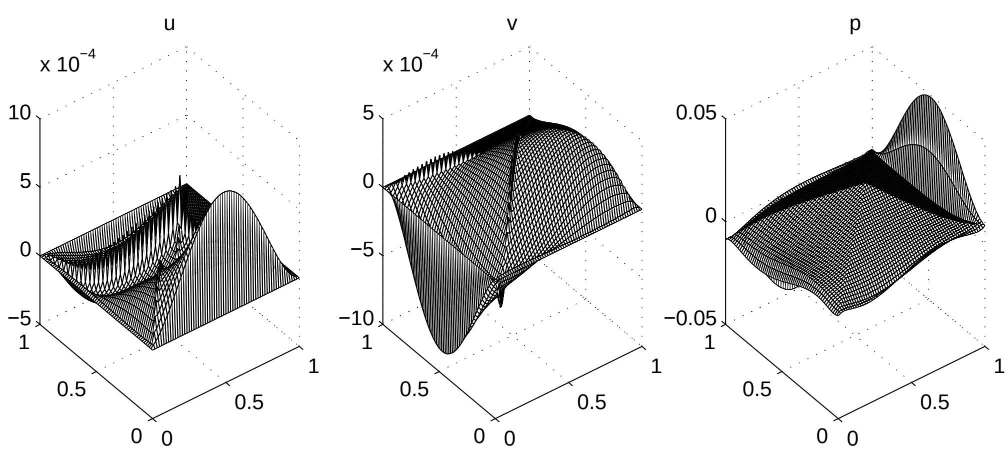


Figure 4.12: Error functions for Algorithm 4.2 with $P_2 - P_1 - P_1$.

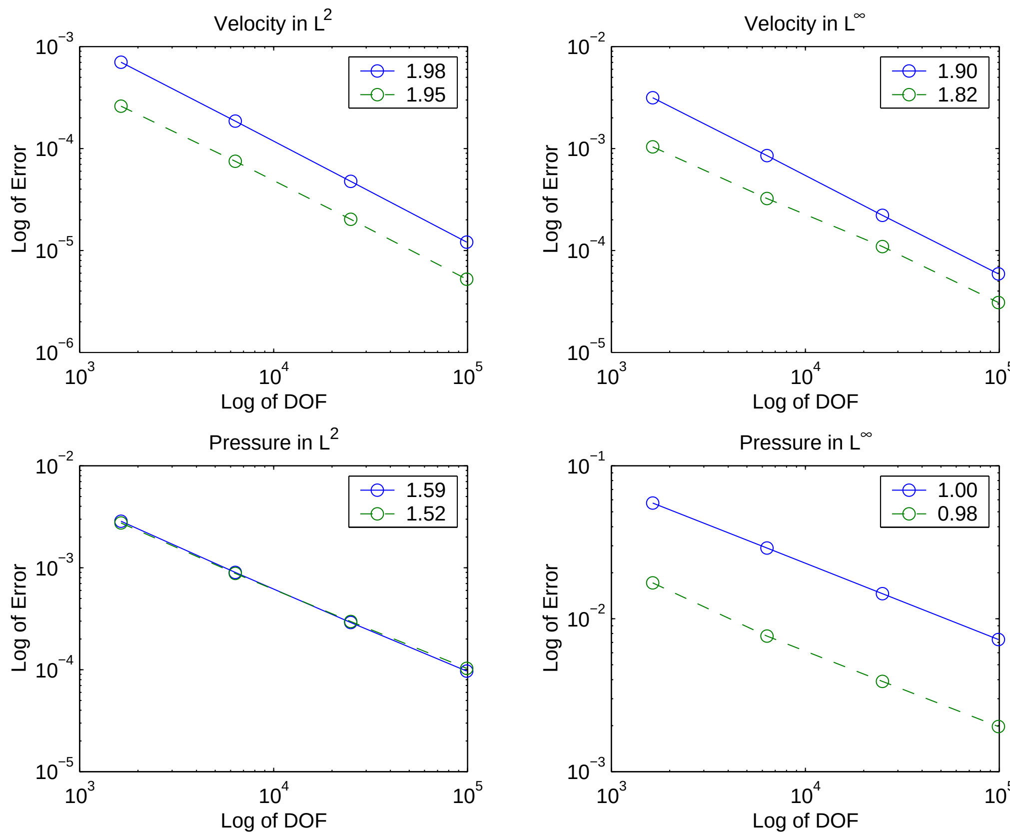


Figure 4.13: Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_2$.

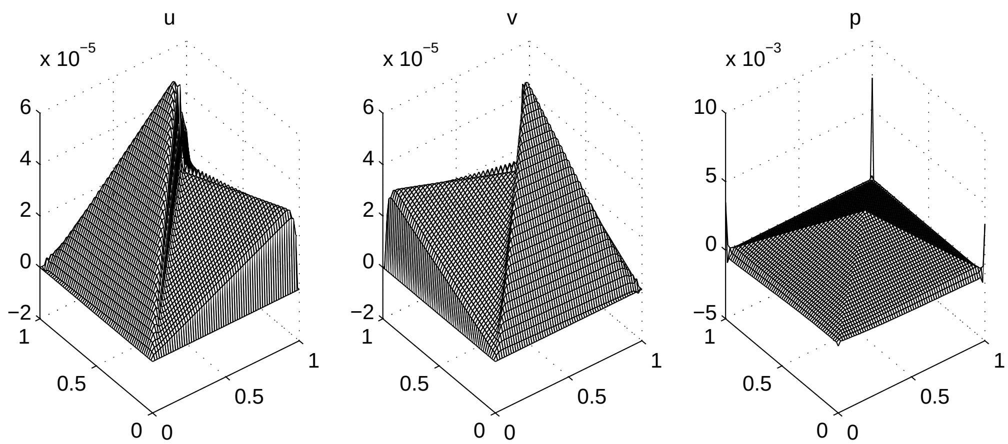


Figure 4.14: Error functions for Algorithm 4.1 with $P_2 - P_1 - P_2$.

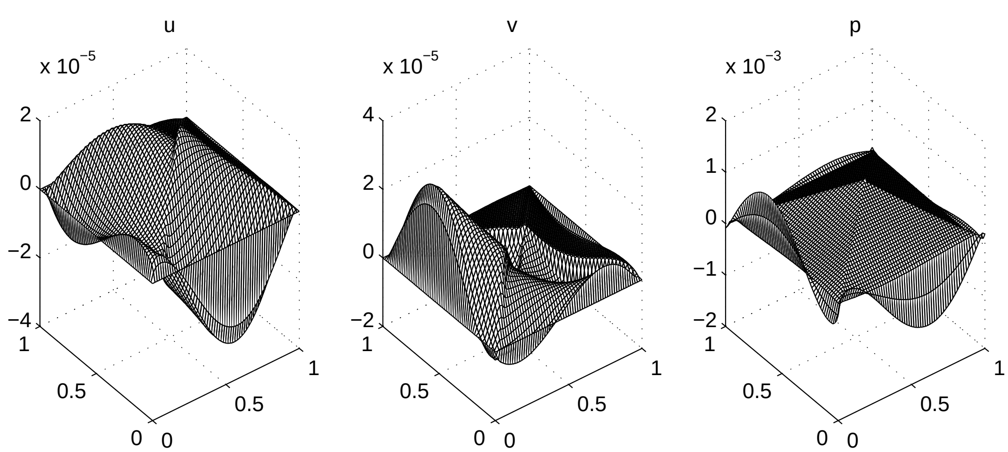


Figure 4.15: Error functions for Algorithm 4.2 with $P_2 - P_1 - P_2$.

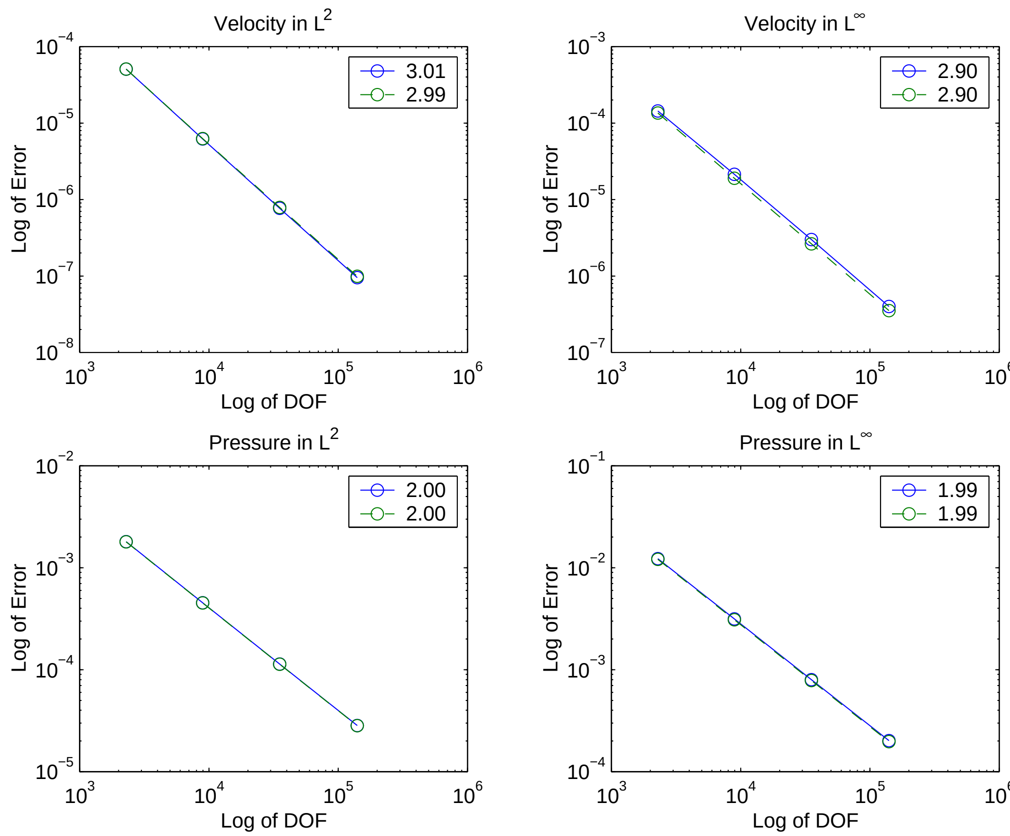


Figure 4.16: Error Decay of of Algorithms 4.1 (Solid) and 4.2 (Dashed) with $P_2 - P_1 - P_3$.

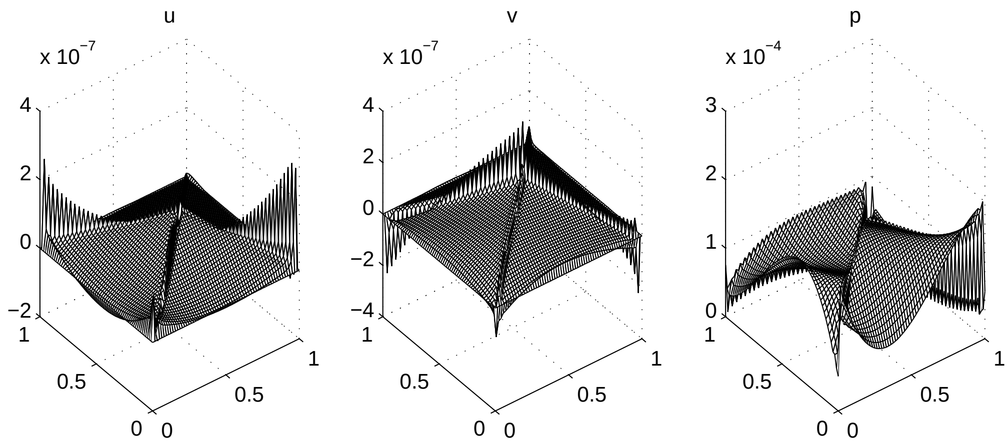


Figure 4.17: Error functions of Algorithm 4.1 with $P_2 - P_1 - P_3$

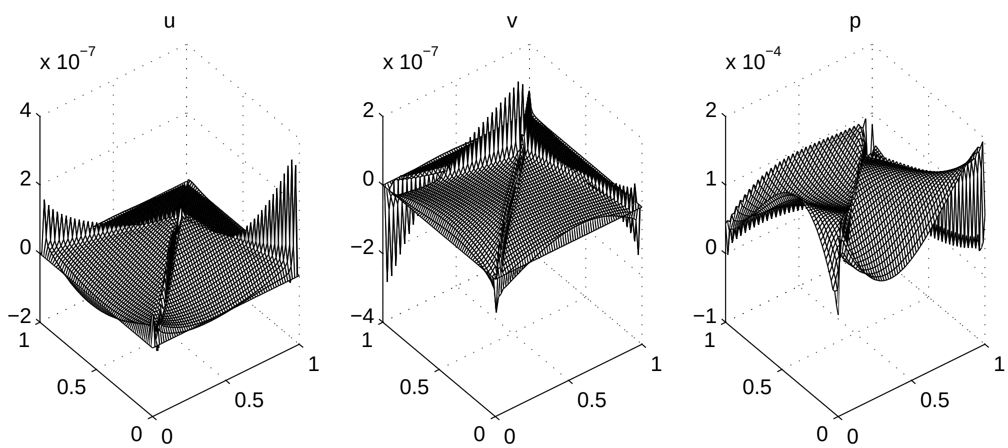


Figure 4.18: Error functions for Algorithm 4.2 with $P_2 - P_1 - P_3$.

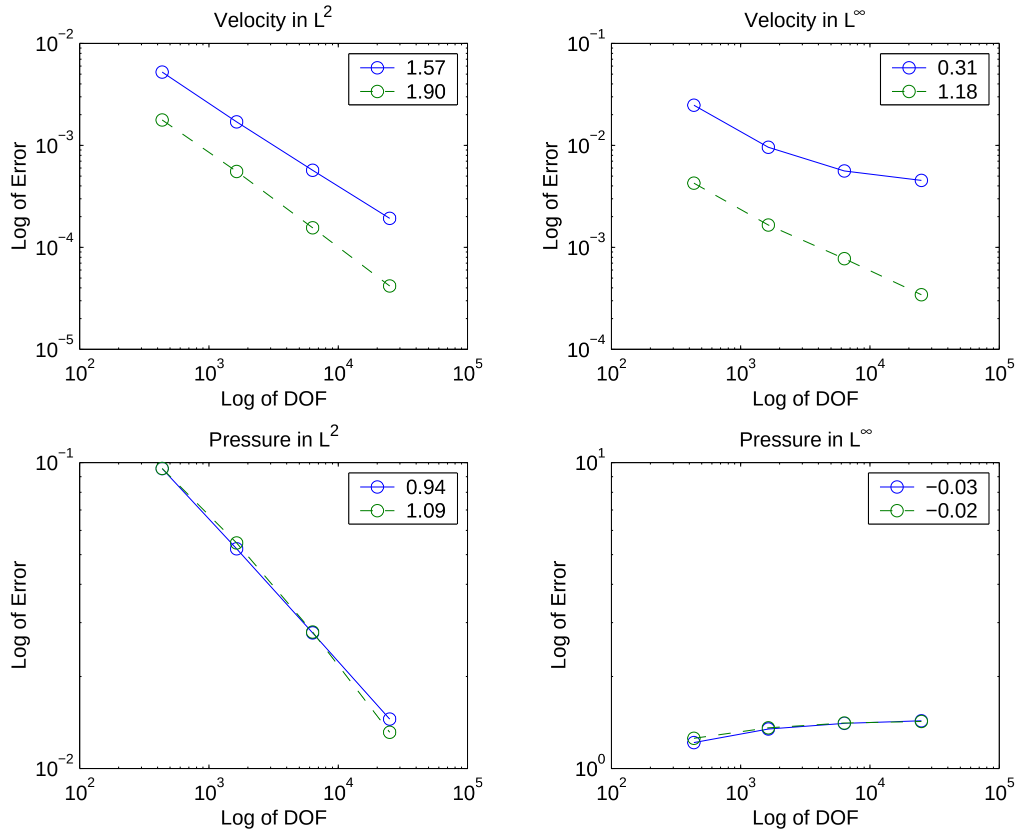


Figure 4.19: Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_1 - P_1 - P_1$.

4.3.2 Numerical Experiments for Algorithms 4.3-4.4

In this subsection we present numerical experiments for Algorithms 4.3-4.4. By the incompatibility condition in Remark 3.3, the pressure for the Algorithms 4.3-4.4 which are imposed Dirichlet condition do not pointwisely converge to exact solution for any finite elements spaces (see Figures 4.19, 4.22, 4.25, and 4.28). Even though the errors of pressure seem to be decreasing in L^2 space for this examples, it may not true for other problems. Also the order of convergence for velocity is not stable even the best combination $P_2 - P_1 - P_3$ (see Figure 4.31).

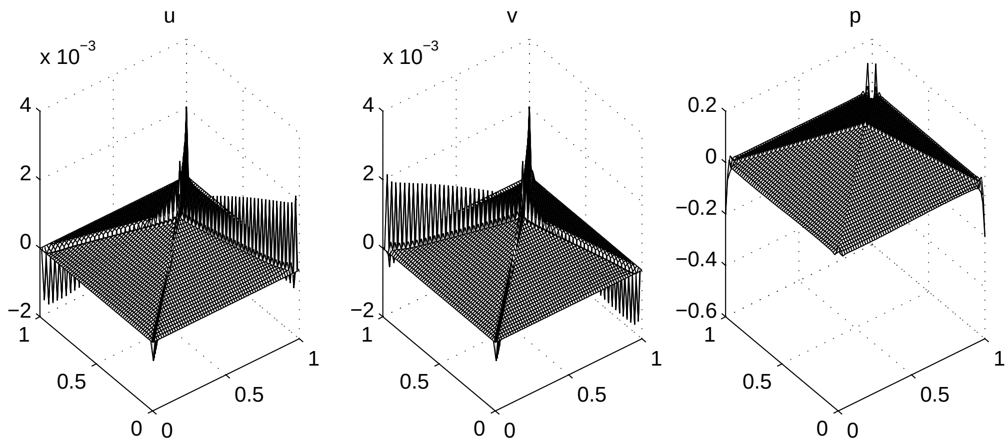


Figure 4.20: Error functions for Algorithm 4.3 with $P_1 - P_1 - P_1$.

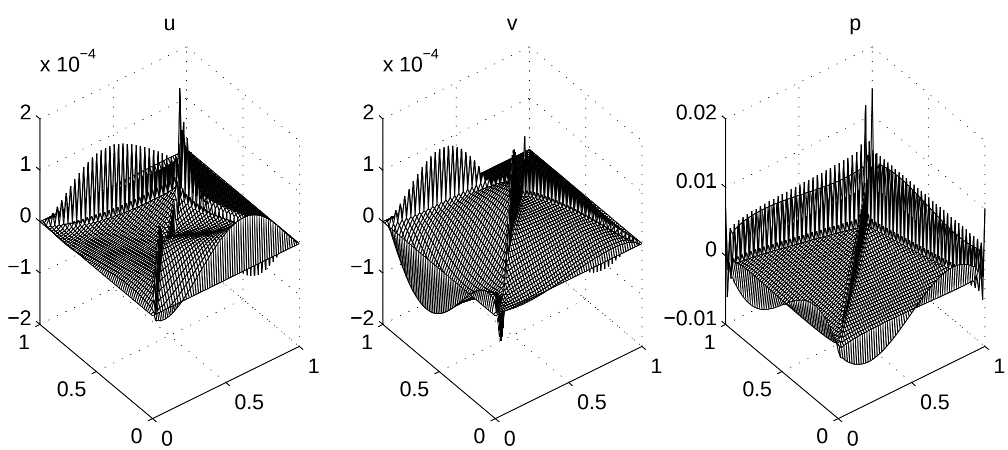


Figure 4.21: Error functions for Algorithm 4.4 with $P_1 - P_1 - P_1$.

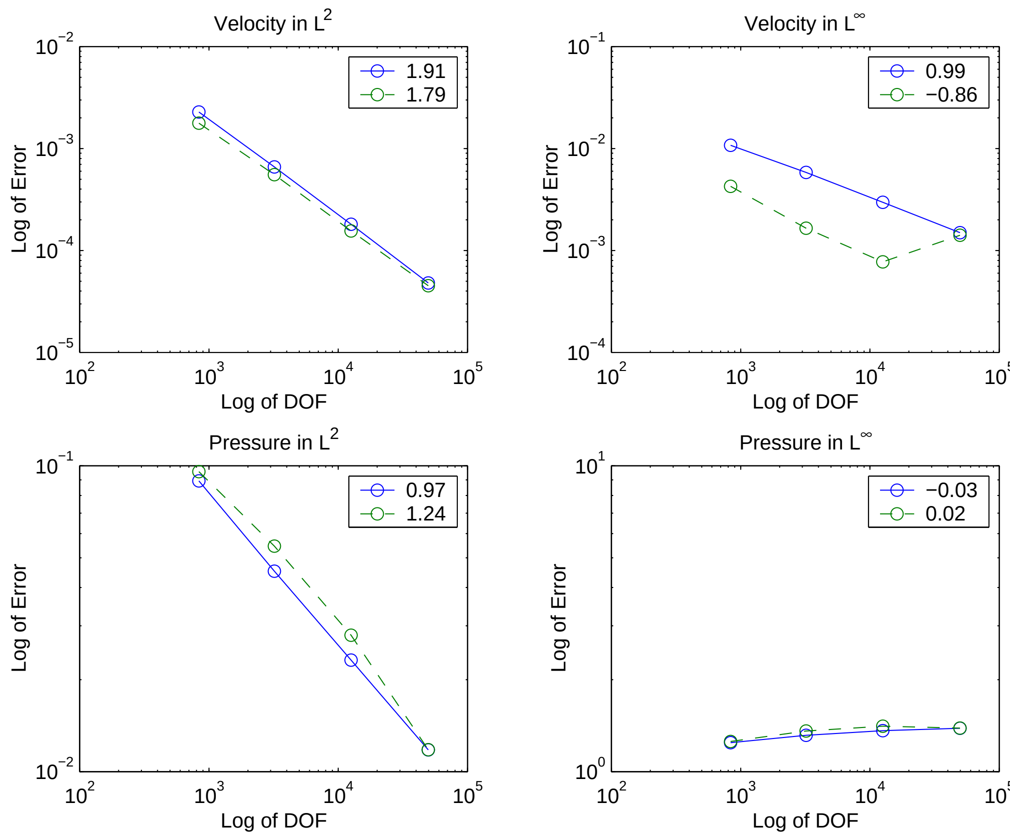


Figure 4.22: Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_1 - P_1 - P_2$

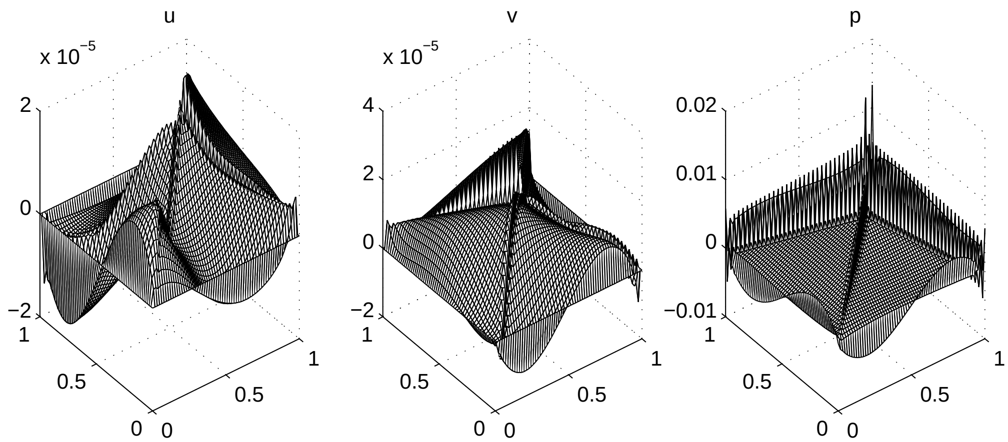


Figure 4.23: Error functions for Algorithm 4.3 with $P_1 - P_1 - P_2$.

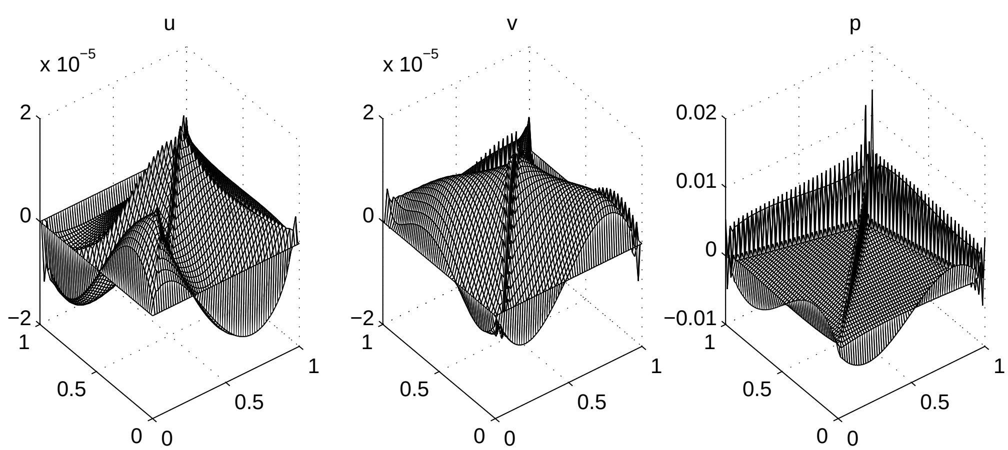


Figure 4.24: Error functions for Algorithm 4.4 with $P_1 - P_1 - P_2$.

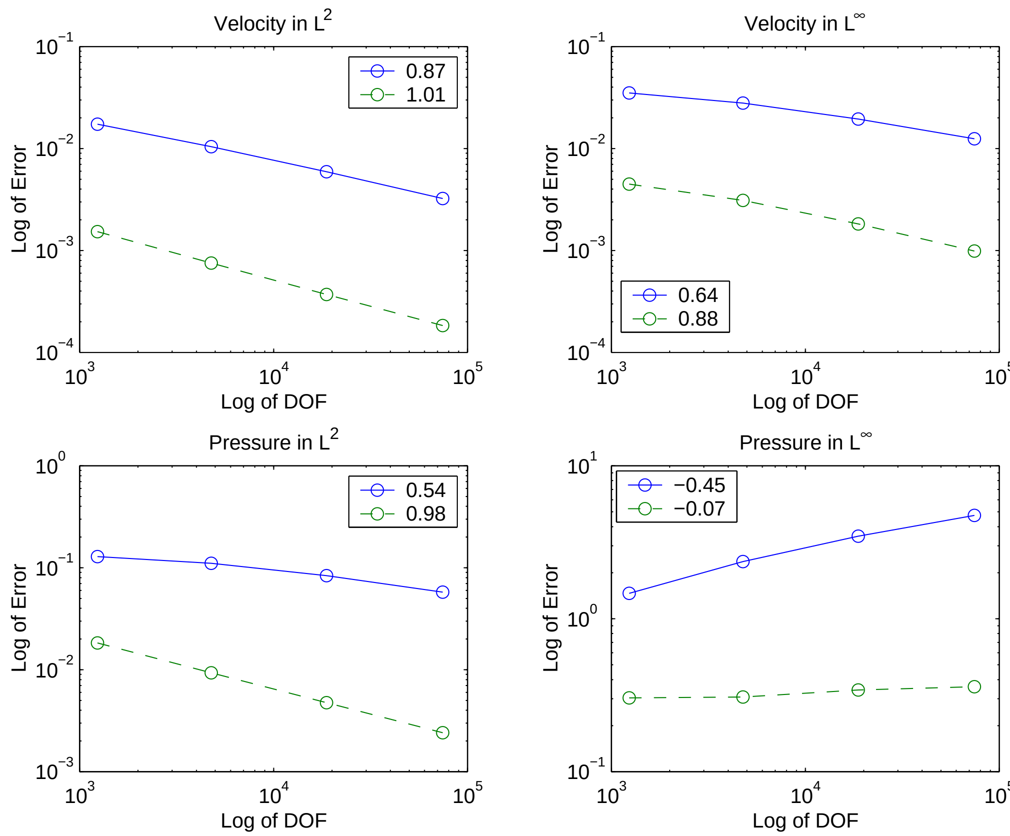


Figure 4.25: Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_1$

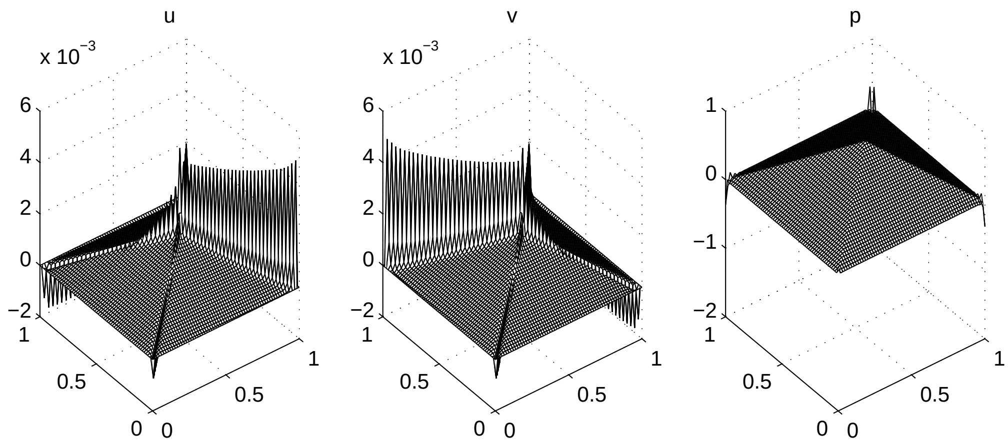


Figure 4.26: Error functions for Algorithm 4.3 with $P_2 - P_1 - P_1$.

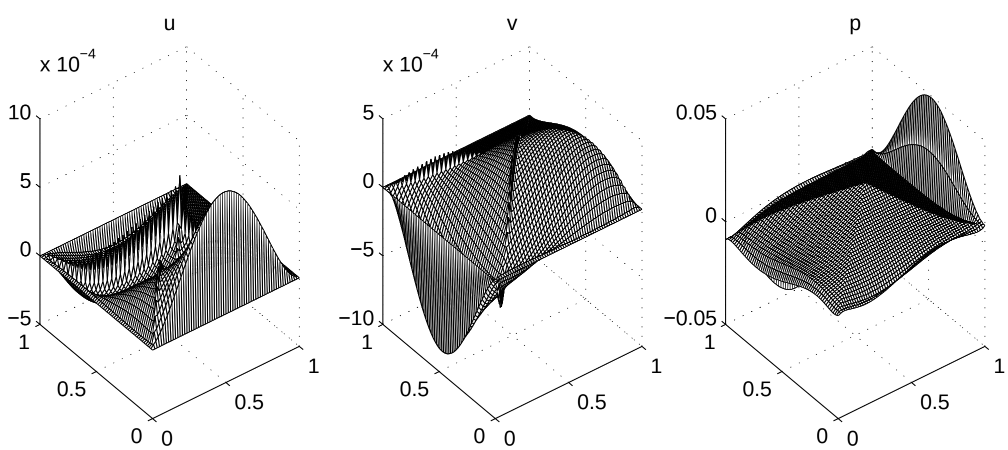


Figure 4.27: Error functions for Algorithm 4.4 with $P_2 - P_1 - P_1$.

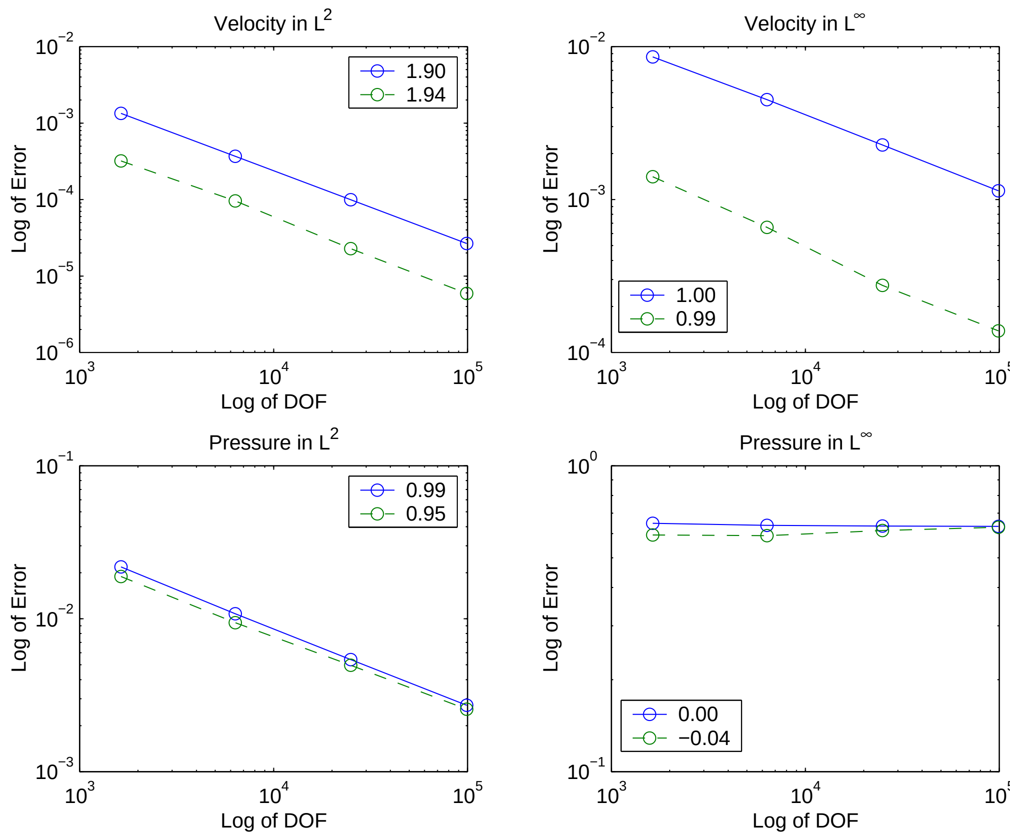


Figure 4.28: Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_2$.

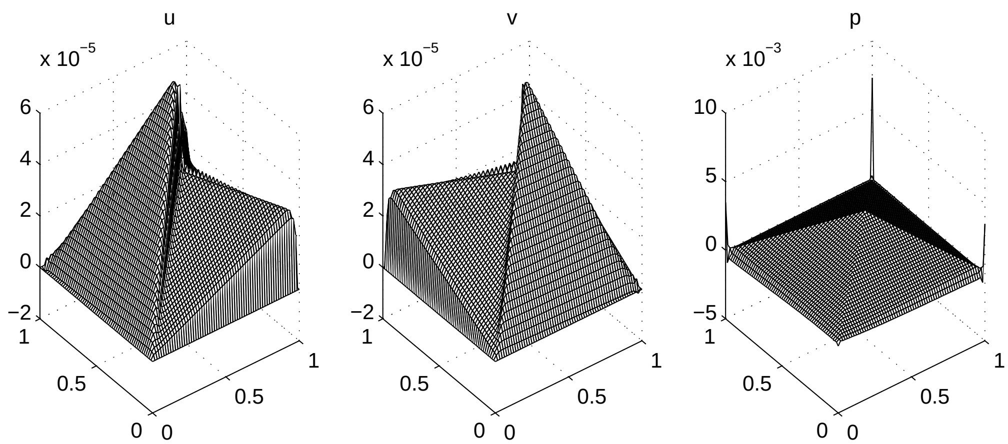


Figure 4.29: Error functions for Algorithm 4.3 with $P_2 - P_1 - P_2$.

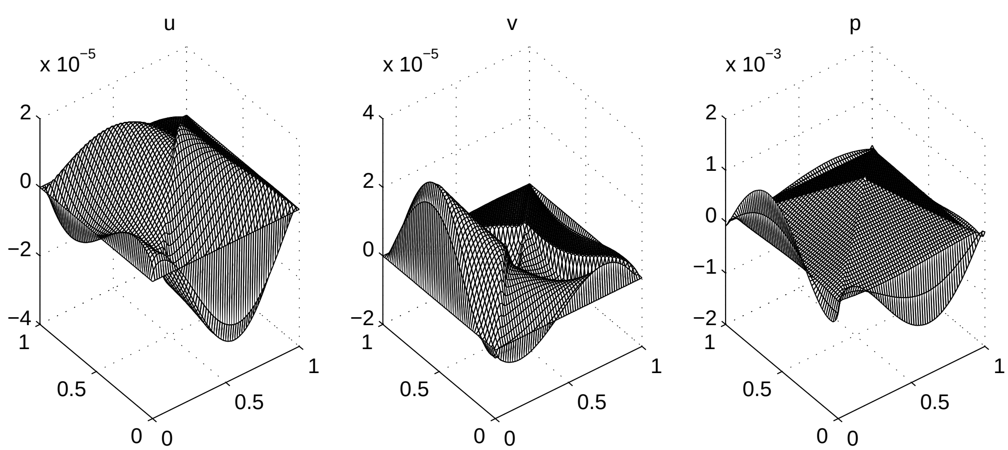


Figure 4.30: Error functions for Algorithm 4.4 with $P_2 - P_1 - P_2$.

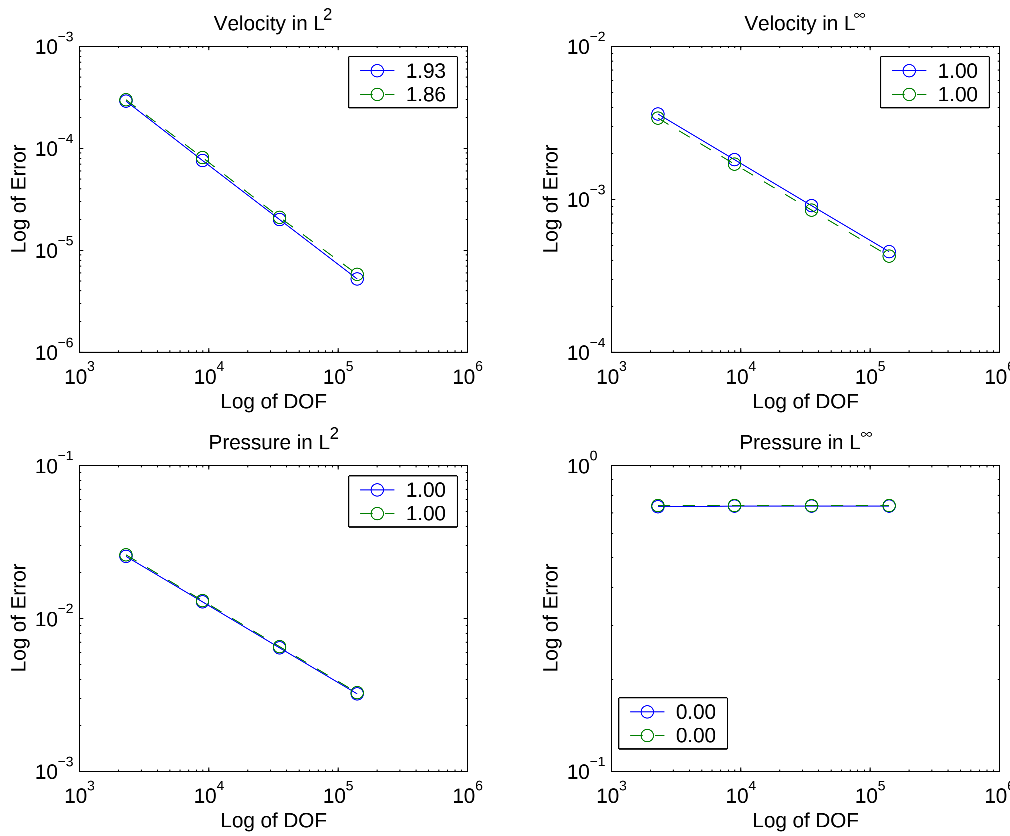


Figure 4.31: Error Decay of of Algorithms 4.3 (Solid) and 4.4 (Dashed) with $P_2 - P_1 - P_3$.

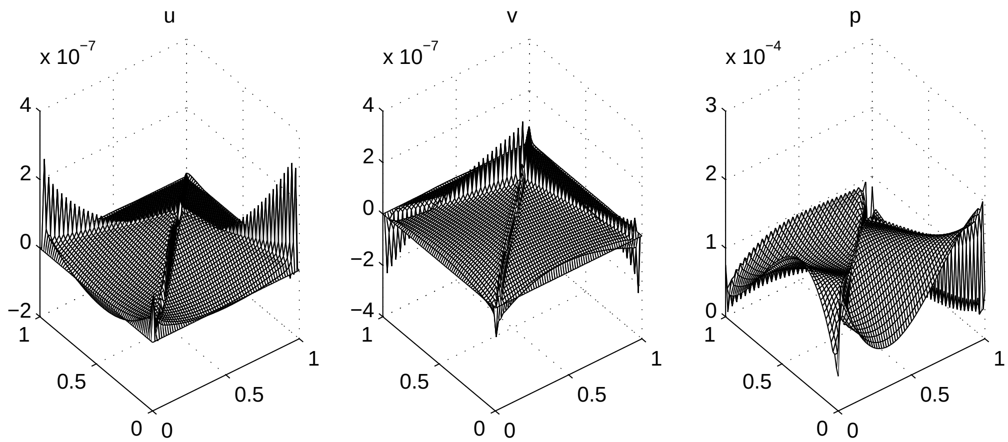


Figure 4.32: Error functions of Algorithm 4.3 with $P_2 - P_1 - P_3$

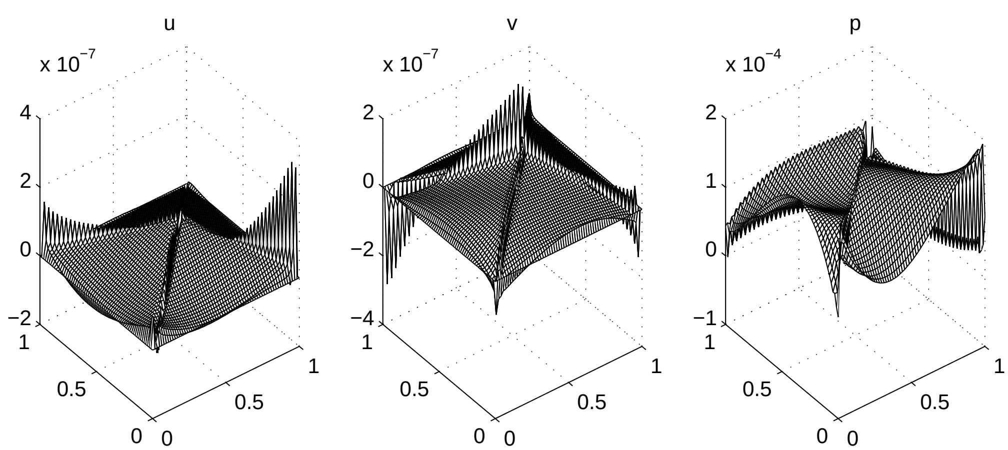


Figure 4.33: Error functions for Algorithm 4.4 with $P_2 - P_1 - P_3$.

4.4 Gauge-Uzawa Method for Stationary Stokes

The Gauge-Uzawa method is a reformulation of Algorithm 4.1 which eliminates the boundary computation. We recall that

$$\begin{cases} -\dfrac{1}{Re}\triangle \mathbf{a}^{n+1} = \mathbf{f}, & \text{in } \Omega, \\ \mathbf{a}^{n+1} = -\nabla\phi^n, & \text{on } \partial\Omega. \end{cases} \tag{4.4.1}$$

We introduce the auxiliary velocity $\widehat{\mathbf{u}}^{n+1}$ defined to be $\mathbf{a}^{n+1} = \widehat{\mathbf{u}}^{n+1} - \nabla\phi^n$, and the momentum becomes

$$\begin{cases} -\dfrac{1}{Re}\triangle \widehat{\mathbf{u}}^{n+1} + \dfrac{1}{Re}\nabla\triangle\phi^n = \mathbf{f}, & \text{in } \Omega, \\ \widehat{\mathbf{u}}^{n+1} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4.2}$$

Since $\dfrac{1}{Re}\triangle\phi^n = p^n$ according to (4.2.10) and (4.2.16), we can rewrite (4.4.2) as follows:

$$\begin{cases} -\dfrac{1}{Re}\triangle \widehat{\mathbf{u}}^{n+1} + \dfrac{1}{Re}\nabla p^n = \mathbf{f}, & \text{in } \Omega, \\ \widehat{\mathbf{u}}^{n+1} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4.3}$$

If we set $\rho^{n+1} = \phi^{n+1} - \phi^n$, then we can write

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1} = \widehat{\mathbf{u}}^{n+1} + \nabla(\phi^{n+1} - \phi^n) = \widehat{\mathbf{u}}^{n+1} + \nabla\rho^{n+1}, \tag{4.4.4}$$

and

$$p^{n+1} = \frac{1}{Re}\triangle\phi^{n+1} = p^n - \operatorname{div}\widehat{\mathbf{u}}^{n+1}. \tag{4.4.5}$$

This allows us to remove the variables $\mathbf{a}^n$ and ϕ^n and reformulation Algorithm 4.1 in terms of $\widehat{\mathbf{u}}^n$ and ρ^n by showing (4.4.3)-(4.4.5). The connection with the Uzawa method, namely (4.4.3) and (4.4.4), is now obvious. The discrete Gauge-Uzawa method now reads as follows.

Algorithm 4.5 *(*Discrete Gauge-Uzawa Method for Steady State Stokes*) Start with* $p_h^0 = 0$*.*

Step 1: Find $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ *such that*

$$\frac{1}{Re}\left\langle \nabla \widehat{\mathbf{u}}_h^{n+1}, \nabla \mathbf{w}_h \right\rangle - \langle p_h^n, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h^0, \tag{4.4.6}$$

Step 2: Find $\rho_h^{n+1} \in \mathbb{R}_h$ *such that*

$$\left\langle \nabla \rho_h^{n+1}, \nabla \psi_h \right\rangle = \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1}, \psi_h \right\rangle \quad \forall \psi_h \ \in \mathbb{R}_h. \tag{4.4.7}$$

Step 3: Update

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}^{n+1} + \nabla \rho_h^{n+1}. \tag{4.4.8}$$

Step 4: Find $p_h^{n+1} \in \mathbb{P}_h$ *such that*

$$\left\langle p_h^{n+1}, q_h \right\rangle = \langle p_h^n, q_h \rangle - \frac{1}{Re}\left\langle \nabla \rho_h^{n+1}, \nabla q_h \right\rangle, \quad \forall q_h \in \mathbb{P}_h. \tag{4.4.9}$$

Remark 4.6 (Boundary Derivatives) We stress that no boundary differentiation is required in Algorithm 4.5. This makes it applicable in any dimension. We also note that, except for Steps 2 and 3 This scheme reduces to Uzawa.

Remark 4.7 (Discrete Divergence Free Velocity) The velocity of Uzawa is not discrete divergence free. However, note that (4.4.7) and (4.4.8) yield

$$\left\langle \mathbf{u}_h^{n+1}, \nabla \psi_h \right\rangle = \left\langle \nabla \rho_h^{n+1}, \nabla \psi_h \right\rangle - \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1}, \psi_h \right\rangle, \quad \forall \psi_h \in \mathbb{R}_h. \tag{4.4.10}$$

Therefore, even though $\mathbf{u}_h^{n+1}$ is discontinuous, its divergence is orthogonal to $\mathbb{R}_h$.

We define $(\mathbf{u}_h, p_h) \in \mathbb{V}_h \times \mathbb{P}_h$ to be the solution of the following discrete Stokes problem:

$$\begin{cases} \dfrac{1}{Re}\langle \nabla \mathbf{u}_h, \nabla \mathbf{w}_h \rangle - \langle p_h, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle, & \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle \mathbf{u}_h, \nabla q_h \rangle = 0, & \forall q_h \in \mathbb{P}_h. \end{cases} \tag{4.4.11}$$

Then we have the following error bound [1, 12].

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h\|\mathbf{u} - \mathbf{u}_h\|_1 + h\|p - p_h\|_0 \le Ch^{\kappa+1}\left(\|\mathbf{u}\|_{\kappa+1} + \|p\|_\kappa\right). \tag{4.4.12}$$

To study the convergence of $(\mathbf{u}_h^n, p_h^n)$ of Algorithm 4.5 to $(\mathbf{u}_h, p_h)$, we introduce the error functions

$$\widehat{\mathbf{E}}_h^{n+1} = \mathbf{u}_h - \widehat{\mathbf{u}}_h^{n+1}, \quad \mathbf{E}_h^{n+1} = \mathbf{u}_h - \mathbf{u}_h^{n+1}, \quad e_h^{n+1} = p_h - p_h^{n+1}. \tag{4.4.13}$$

The error functions $\widehat{\mathbf{E}}_h^{n+1}$, $\mathbf{E}_h^{n+1}$ have the following properties:

$$\widehat{\mathbf{E}}_h^{n+1} \in \mathbb{V}_h, \tag{4.4.14}$$

and

$$\widehat{\mathbf{E}}_h^{n+1}\mathbf{u}_h - \widehat{\mathbf{u}}_h^{n+1} = \mathbf{u}_h - \mathbf{u}_h^{n+1} + \nabla\rho_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla\rho_h^{n+1}. \tag{4.4.15}$$

Furthermore, if $\mathbb{P}_h \subseteq \mathbb{R}_h$, then

$$\left\langle \mathbf{E}_h^{n+1}, \nabla q_h \right\rangle = 0, \quad \forall q_h \in \mathbb{P}_h. \tag{4.4.16}$$

The restriction $\mathbb{P}_h \subseteq \mathbb{R}_h$ means that we are dealing with continuous pressures. Since the inf-sup constant satisfies $0 < \beta \le 1$, by Lemma 1.11, the following theorem shows convergence of Algorithm 4.5 provided the inf-sup condition holds and $\mathbb{P}_h \subseteq \mathbb{R}_h$.

Theorem 4.1 *Let the inf-sup Assumption 4 hold and let the pressure space* $\mathbb{P}_h$ *and gauge variable space* $\mathbb{R}_h$ *satisfy* $\mathbb{P}_h \subseteq \mathbb{R}_h$. *Let* $\beta \le 1$ *be the inf-sup constant in Assumption 4 Then we have, for all iteration step* n,

$$\frac{1}{Re^2}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2 + \|e_h^{n+1}\|_0^2 \le \|e_h^n\|_0^2, \tag{4.4.17}$$

$$\|e_h^n\|_0 \le \frac{1}{\beta Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0, \tag{4.4.18}$$

$$\|e_h^{n+1}\|_0 \le \sqrt{1-\beta^2}\|e_h^n\|_0. \tag{4.4.19}$$

Proof. By subtracting (4.4.6) from (4.4.11), we have

$$\frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}_h^{n+1}, \nabla\mathbf{w}_h \right\rangle - \langle e_h^n, \operatorname{div}\mathbf{w}_h\rangle = 0, \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \tag{4.4.20}$$

Now, we choose $\mathbf{w}_h = \widehat{\mathbf{E}}_h^{n+1}$. Since $e_h^{n+1} \subseteq C^0(\Omega)$ by definition of $\mathbb{P}_h$ in (1.2.38), using (4.4.15), we get

$$\begin{aligned} \frac{1}{Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2 &= -\left\langle \nabla e_h^n, \widehat{\mathbf{E}}_h^{n+1}\right\rangle \\ &= -\left\langle \nabla e_h^n, \mathbf{E}_h^{n+1} + \nabla\rho_h^{n+1}\right\rangle. \end{aligned} \tag{4.4.21}$$

By (4.4.16), we have $\left\langle \nabla e_h^n, \mathbf{E}_h^{n+1}\right\rangle = 0$. and, by (4.4.9), we deduce

$$\begin{aligned} \frac{1}{Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2 &= -\left\langle \nabla e_h^n, \nabla\rho_h^{n+1}\right\rangle \\ &= Re\left\langle e_h^n, p_h^{n+1} - p_h^n\right\rangle \\ &= -Re\left\langle e_h^n, e_h^{n+1} - e_h^n\right\rangle \\ &= -\frac{Re}{2}\left(\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2 - \|e_h^{n+1} - e_h^n\|_0^2\right). \end{aligned} \tag{4.4.22}$$

Thus we obtain

$$\frac{1}{Re^2}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2 + \frac{1}{2}\|e_h^{n+1}\|_0^2 = \frac{1}{2}\|e_h^n\|_0^2 + \frac{1}{2}\|e_h^{n+1} - e_h^n\|_0^2. \tag{4.4.23}$$

Now we estimate $\|e_h^{n+1} - e_h^n\|_0^2$ in (4.4.23), using (4.4.6)-(4.4.11) as follows:

$$\begin{aligned} \|e_h^{n+1} - e_h^n\|_0^2 &= \left\langle p_h^{n+1} - p_h^n, p_h^{n+1} - p_h^n\right\rangle \\ &= -\frac{1}{Re}\left\langle \nabla\rho_h^{n+1}, \nabla(p_h^{n+1} - p_h^n)\right\rangle \\ &= \frac{1}{Re}\left\langle \operatorname{div}\widehat{\mathbf{E}}_h^{n+1}, e_h^{n+1} - e_h^n\right\rangle \\ &\leq \frac{1}{Re}\|\operatorname{div}\widehat{\mathbf{E}}_h^{n+1}\|_0\|e_h^{n+1} - e_h^n\|_0. \end{aligned} \tag{4.4.24}$$

Since we have $\|\operatorname{div}\widehat{\mathbf{E}}_h^{n+1}\|_0 \leq \|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0$ by Lemma 1.9, (4.4.24) becomes

$$\|e_h^{n+1} - e_h^n\|_0^2 \leq \frac{1}{Re^2}\|\operatorname{div}\widehat{\mathbf{E}}_h^{n+1}\|_0^2 \leq \frac{1}{Re^2}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2. \tag{4.4.25}$$

So (4.4.23) changes into

$$\frac{1}{Re^2}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0^2 + \|e_h^{n+1}\|_0^2 \le \|e_h^n\|_0^2, \tag{4.4.26}$$

which is (4.4.17). Now we prove (4.4.18). Since $e_h^n \in \mathbb{P}_h$, by Assumption 4, there exists a function $\mathbf{v}_h \in \mathbb{V}_h$ such that

$$\langle \text{div } \mathbf{v}_h \, , \, e_h^n \rangle = \|e_h^n\|_0^2 \quad \text{and} \quad \|\nabla \mathbf{v}_h\|_0 \le \frac{1}{\beta}\|e_h^n\|_0. \tag{4.4.27}$$

By (4.4.20) and (4.4.27), we get

$$\begin{aligned} \|e_h^n\|_0^2 &= \langle \text{div } \mathbf{v}_h \, , \, e_h^n \rangle \\ &= \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}_h^{n+1} \, , \, \nabla \mathbf{v}_h \right\rangle \\ &= \frac{1}{Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0 \|\nabla \mathbf{v}_h\|_0 \\ &\le \frac{1}{\beta Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0 \|e_h^n\|_0, \end{aligned} \tag{4.4.28}$$

whence

$$\|e_h^n\|_0 \le \frac{1}{\beta Re}\|\nabla\widehat{\mathbf{E}}_h^{n+1}\|_0. \tag{4.4.29}$$

So (4.4.18) is proved. Finally, by (4.4.17) and (4.4.18), we get easily

$$\beta^2\|e_h^n\|_0^2 + \|e_h^{n+1}\|_0^2 \le \|e_h^n\|_0^2 \tag{4.4.30}$$

or equivalently

$$\|e_h^{n+1}\|_0^2 \le (1-\beta^2)\|e_h^n\|_0^2. \tag{4.4.31}$$

This completes the proof. ∎

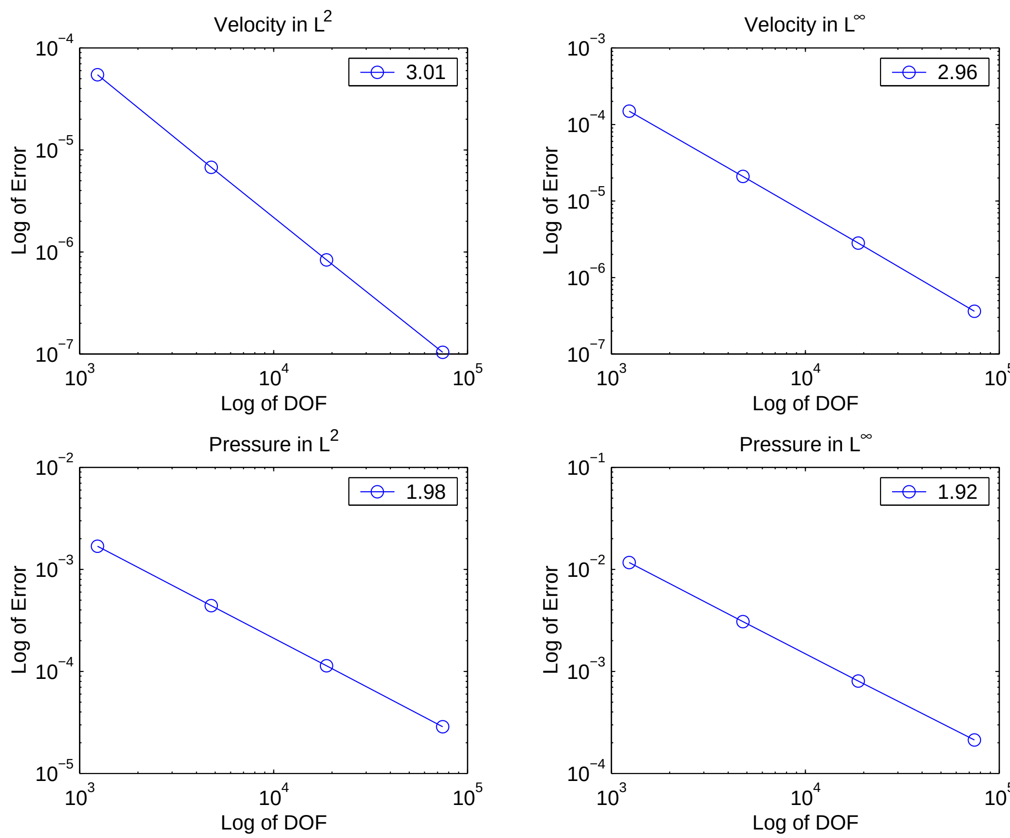


Figure 4.34: Mesh Analysis of Algorithm 4.5 with Spaces $P2 - P1 - P1$

4.5 Numerical Experiment for Gauge-Uzawa

We examine Algorithm 4.5 for Example 1.3.1 and report the results in Figure 4.34 and 4.35. Compared with Figures 4.4-4.33, the errors of Algorithm 4.5 are much smaller than those of gauge methods. Because Algorithm 4.5 does not deal with boundary values, we can conclude that the relatively big errors of gauge Algorithms are caused by the boundary approximation.

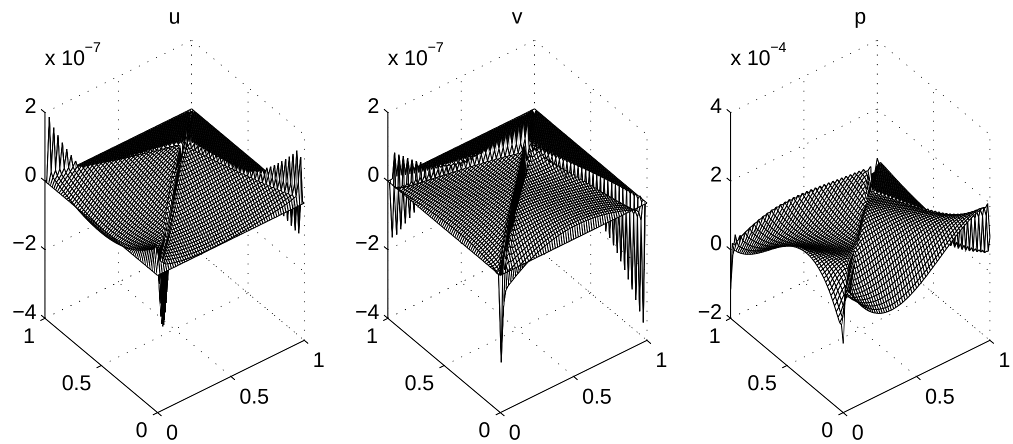


Figure 4.35: Error Functions of Algorithm 4.5 with Spaces $P2 - P1 - P1$

4.6 Uzawa Method

The following Uzawa algorithm is a well known iteration solver of the Stokes system (4.0.1) (see for instance [1, 9, 12]):

Algorithm 4.6 *(*Uzawa Method*) Begin with an initial guess* p^0

step 1: *Find* $\mathbf{u}^{n+1}$ *as the solution of*

$$\begin{cases} -\dfrac{1}{Re}\triangle \mathbf{u}^{n+1} + \nabla p^n = \mathbf{f}, & \textit{in } \Omega, \\ \mathbf{u}^{n+1} = 0, & \textit{on } \partial\Omega. \end{cases} \tag{4.6.1}$$

step 2: *Find* p^{n+1}

$$p^{n+1} = p^n - \frac{\alpha}{Re}\mathrm{div}\, \mathbf{u}^{n+1}, \quad \textit{where } 0 < \alpha < 1. \tag{4.6.2}$$

The convergence of Uzawa Method 4.6 is already proved by using boundedness and coercivity of the Schur complement operator [1], provided $0 < \alpha < 1$ is sufficiently small. The main goal in this section is to find more extended convergence rate and optimal α for Uzawa algorithm.

If we compare Gauge-Uzawa and Uzawa Algorithms 4.5 and 4.6 under the assumptions $\mathbb{R}_h = \mathbb{P}_h$ and $\alpha = 1$, we can see easily that the velocity $\mathbf{u}^{n+1}$ in Algorithm 4.6 corresponds to $\widehat{\mathbf{u}}^{n+1}$ in Algorithm 4.5. So Gauge-Uzawa Algorithm 4.5 may be viewed as a Uzawa Algorithm with a projection step (4.4.8) into a divergence free space. So Theorem 4.1 provides a proof of convergence of Uzawa Algorithm 4.6 provided $\mathbb{R}_h = \mathbb{P}_h$ and $\alpha = 1$.

We now investigate the convergence rate and optimal choice of α for the discrete Uzawa Algorithm 4.7 below. Since the velocity $\mathbf{u}^{n+1}$ in Uzawa method is not divergence free, the pressure space $\mathbb{P}_h$ does not need to be a subspace of $\mathbf{C}^0(\Omega)$. We thus $\mathbb{P}_h$ to be a subspace in $L^2(\Omega)$:

$$\mathbb{P}_h = \{p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}(K), \quad \forall K \in \mathfrak{T}, \quad \int_\Omega p_h d\Omega = 0\}; \tag{4.6.3}$$

$\mathcal{P}(K)$ are spaces of uniformly bounded degree polynomials with respect to $K \in \mathfrak{T}$.

Algorithm 4.7 *(*Discrete Uzawa Method*) Begin with initial guess* $p_h^0 \in \mathbb{P}_h$

step 1: *Find* $\mathbf{u}_h^{n+1} \in \mathbb{V}_h^0$ *as the solution of*

$$\frac{1}{Re}\left\langle \nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{w}_h \right\rangle - \langle p_h^n, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h^0. \tag{4.6.4}$$

step 2: *Find* $p_h^{n+1} \in \mathbb{P}_h$ *such that*

$$\left\langle p_h^{n+1}, q_h \right\rangle = \langle p_h^n, q_h \rangle - \frac{\alpha}{Re}\left\langle \operatorname{div} \mathbf{u}_h^{n+1}, q_h \right\rangle, \quad \forall q_h \in \mathbb{P}_h. \tag{4.6.5}$$

We use the following error notations

$$\mathbf{E}_h^{n+1} = \mathbf{u}_h - \mathbf{u}_h^{n+1} \quad \text{and} \quad e_h^{n+1} = p_h - p_h^{n+1}, \tag{4.6.6}$$

where $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ is a solution of Algorithm 4.7 and $(\mathbf{u}_h, p_h)$ is a solution of discrete saddle point formulation (4.4.11). We note that $\left\langle \mathbf{u}_h^{n+1}, \nabla q_h \right\rangle$ for $q_h \in \mathbb{P}_h$

does not make sense in this context and that $\left\langle \operatorname{div} \mathbf{u}_h^{n+1}, q_h \right\rangle \neq 0$ for $q_h \in \mathbb{P}_h$ in general.

Theorem 4.2 *Let the inf-sup Assumption 4 hold with constant β. Then we have, for all iteration steps n,*

$$\frac{2\alpha - \alpha^2}{Re^2} \left\| \nabla \mathbf{E}_h^{n+1} \right\|_0^2 + \left\| e_h^{n+1} \right\|_0^2 \leq \| e_h^n \|_0^2 \tag{4.6.7}$$

$$\left\| e_h^{n+1} \right\|_0^2 \leq (1 - 2\alpha\beta^2 + \alpha^2\beta^2) \| e_h^n \|_0^2. \tag{4.6.8}$$

Proof. By subtracting (4.6.4) from (4.4.11), we have

$$\frac{1}{Re} \left\langle \nabla \mathbf{E}_h^{n+1}, \nabla \mathbf{w}_h \right\rangle - \langle e_h^n, \operatorname{div} \mathbf{w}_h \rangle = 0, \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \tag{4.6.9}$$

Now, we choose $\mathbf{w}_h = \mathbf{E}_h^{n+1}$. Then (4.6.9) becomes

$$\frac{1}{Re} \left\| \nabla \mathbf{E}_h^{n+1} \right\|_0^2 - \left\langle e_h^n, \operatorname{div} \mathbf{E}_h^{n+1} \right\rangle = 0. \tag{4.6.10}$$

By (4.6.5) and (4.6.10),

$$\begin{aligned} \frac{1}{Re} \left\| \nabla \mathbf{E}_h^{n+1} \right\|_0^2 &= -\left\langle e_h^n, \operatorname{div} \mathbf{u}_h^{n+1} \right\rangle \\ &= \frac{Re}{\alpha} \left\langle e_h^n, p_h^{n+1} - p_h^n \right\rangle \\ &= -\frac{Re}{\alpha} \left\langle e_h^n, e_h^{n+1} - e_h^n \right\rangle \\ &= -\frac{Re}{2\alpha} \left(\left\| e_h^{n+1} \right\|_0^2 - \| e_h^n \|_0^2 - \left\| e_h^{n+1} - e_h^n \right\|_0^2 \right). \end{aligned} \tag{4.6.11}$$

Thus we have

$$\frac{2\alpha}{Re^2} \left\| \nabla \mathbf{E}_h^{n+1} \right\|_0^2 + \left\| e_h^{n+1} \right\|_0^2 = \| e_h^n \|_0^2 + \left\| e_h^{n+1} - e_h^n \right\|_0^2. \tag{4.6.12}$$

Now we estimate $\left\| e_h^{n+1} - e_h^n \right\|_0^2$. Arguing as in (4.4.24), we get

$$\left\| e_h^{n+1} - e_h^n \right\|_0^2 \leq \frac{\alpha}{Re} \left\| \operatorname{div} \mathbf{E}_h^{n+1} \right\|_0 \left\| e_h^{n+1} - e_h^n \right\|_0, \tag{4.6.13}$$

and

$$\left\|e_h^{n+1} - e_h^n\right\|_0^2 \leq \frac{\alpha^2}{Re^2}\left\|\text{div } \mathbf{E}_h^{n+1}\right\|_0^2 \leq \frac{\alpha^2}{Re^2}\|\nabla \mathbf{E}_h\|_0^2. \tag{4.6.14}$$

So (4.6.11) becomes

$$\frac{2\alpha - \alpha^2}{Re^2}\left\|\nabla \mathbf{E}_h^{n+1}\right\|_0^2 + \left\|e_h^{n+1}\right\|_0^2 \leq \|e_h^n\|_0^2, \tag{4.6.15}$$

which implies (4.6.7).

Now we prove (4.6.8). Since $e_h^n \in \mathbb{P}_h$, reasoning as in (4.4.28), we obtain

$$\|e_h^n\|_0^2 \leq \frac{1}{\beta Re}\left\|\nabla \mathbf{E}_h^{n+1}\right\|_0 \|e_h^n\|_0, \tag{4.6.16}$$

or equivalently

$$\|e_h^n\|_0^2 \leq \frac{1}{\beta^2 Re^2}\left\|\nabla \mathbf{E}_h^{n+1}\right\|_0^2. \tag{4.6.17}$$

By (4.6.7) and (4.6.17), we get easily

$$\beta^2(2\alpha - \alpha^2)\|e_h^n\|_0^2 + \left\|e_h^{n+1}\right\|_0^2 \leq \|e_h^n\|_0^2, \tag{4.6.18}$$

whence

$$\left\|e_h^{n+1}\right\|_0^2 \leq (1 - 2\alpha\beta^2 + \alpha^2\beta^2)\|e_h^n\|_0^2. \tag{4.6.19}$$

Finally the proof is complete. ∎

Remark 4.8 Consider the function

$$f(\alpha) = 1 - 2\alpha\beta^2 + \alpha^2\beta^2. \tag{4.6.20}$$

Since the minimum of $f(\alpha)$ is $1 - \beta^2$ at $\alpha = 1$, we conclude that the optimal value of α is

$$\alpha = 1. \tag{4.6.21}$$

in this analysis. We observe that this results is independent of the domain Ω whereas the eigenvalues of the Schur complement operator, the discrete version of $\mathbf{S} = -\mathrm{div}\,(-\triangle)^{-1}\nabla$, depend on Ω. It is plausible that for a given Ω and finite element space $(\mathbb{V}_h, \mathbb{P}_h)$, a special analysis would yield a better value for α since Uzawa is simply a Richardson iteration for the Schur complement. It is also plausible that for a rectangular domain with high aspect ratio, $\alpha = 1$ is the only choice valid for all aspect ratios. This deserves further investigation.

Chapter 5

Gauge-Uzawa Method for the Navier-Stokes Equations

The gauge methods of Chapter 3 impose boundary conditions on the non-physical variable ϕ, which is smoother than pressure. This advantage of the semidiscretization in time is not fully maintained by space discretization because boundary differentiation, as well as interior differentiation to compute pressure, are numerically unstable procedures. Based on the Gauge-Uzawa method for stationary Stokes equations of Section 4.4, we construct in this chapter the fully discrete Gauge-Uzawa method for the evolution Navier-Stokes equations (1.1.1). This method overcomes the difficulties of gauge methods but retain their advantages.

5.1 Motivation of Gauge-Uzawa Method

The Gauge-Uzawa method (GU) is constructed from Algorithm 3.1 by the change of variable

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n} \tag{5.1.1}$$

which was already defined in (3.3.1). Then the momentum equation (3.2.1) in gauge Algorithm 3.1 becomes

$$\frac{\widehat{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\triangle t}+(\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1}-\frac{1}{Re}\triangle\widehat{\mathbf{u}}^{n+1}+\frac{1}{Re}\nabla\triangle\phi^n=\mathbf{f}(t_{n+1}). \tag{5.1.2}$$

We note that $\widehat{\mathbf{u}}^{n+1}=0$ on $\partial\Omega$ according to Lemma 3.4, and that (5.1.2)correspond to an unconditionally scheme because of the semi-implicit treatment of convection. We introduce new variables s^{n+1} and ρ^{n+1} to treat the higher order term $\nabla\triangle\phi^n$ in (5.1.2),

$$s^{n+1}=\triangle\phi^{n+1}=-\text{div}\,\mathbf{a}^{n+1}=\triangle\phi^n-\text{div}\,\widehat{\mathbf{u}}^{n+1}=s^n-\text{div}\,\widehat{\mathbf{u}}^{n+1} \tag{5.1.3}$$

and

$$\rho^{n+1}=\phi^{n+1}-\phi^n. \tag{5.1.4}$$

Then we can formulate the Gauge-Uzawa method as follows:

Algorithm 5.1 (Time Discrete Gauge-Uzawa Method) *Start with initial value* $s^0=0$, *and with* $\mathbf{u}^0=\mathbf{u}(\mathbf{x},0)$.

Step 1: *Find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{cases}\dfrac{\widehat{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\triangle t}+(\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1}-\dfrac{1}{Re}\triangle\widehat{\mathbf{u}}^{n+1}+\dfrac{1}{Re}\nabla s^n=\mathbf{f}(t_{n+1}), & \text{in } \Omega,\\ \widehat{\mathbf{u}}^{n+1}=0, & \text{on } \partial\Omega.\end{cases} \tag{5.1.5}$$

Step 2: *Find* ρ^{n+1} *as the solution of*

$$\begin{cases}-\triangle\rho^{n+1}=\text{div}\,\widehat{\mathbf{u}}^{n+1}, & \text{in } \Omega,\\ \dfrac{\partial\rho^{n+1}}{\partial\nu}=0, & \text{on } \partial\Omega.\end{cases} \tag{5.1.6}$$

Step 3: *Update* s^{n+1}

$$s^{n+1}=s^n-\text{div}\,\widehat{\mathbf{u}}^{n+1}. \tag{5.1.7}$$

Step 4: *Update* $\mathbf{u}^{n+1}$

$$\mathbf{u}^{n+1} = \widehat{\mathbf{u}}^{n+1} + \nabla \rho^{n+1}. \tag{5.1.8}$$

If necessary, compute pressure p^{n+1} *as follows:*

$$p^{n+1} = -\frac{\rho^{n+1}}{\triangle t} + \frac{1}{Re} s^{n+1}. \tag{5.1.9}$$

If we consider discrete spaces $\mathbb{V}_h$ and $\mathbb{P}_h$ with polynomial degree at least $m+1$ and m so that Assumption 4 is valid, we can define the full discretization via the Gauge-Uzawa method:

Algorithm 5.2 *(*Fully Discrete Gauge-Uzawa Method*) Start with initial values* $s_h^0 = 0$ *and the given* $\mathbf{u}_h^0$ *in Assumption 7.*

Step 1: *Compute* $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$ *as the solution of*

$$\begin{aligned} &\frac{\left\langle \widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n \,,\, \mathbf{w}_h \right\rangle}{\triangle t} + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \frac{1}{Re} \left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle \\ &-\frac{1}{Re} \left\langle s_h^n \,,\, \operatorname{div} \mathbf{w}_h \right\rangle = \left\langle \mathbf{f}(t_{n+1}) \,,\, \mathbf{w}_h \right\rangle, \qquad \forall \mathbf{w}_h \in \mathbb{V}_h, \end{aligned} \tag{5.1.10}$$

Step 2: *Compute* $\rho_h^{n+1} \in \mathbb{P}_h$ *as the solution of*

$$\left\langle \nabla \rho_h^{n+1} \,,\, \nabla \psi_h \right\rangle = \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \,,\, \psi_h \right\rangle, \quad \forall \psi_h \in \mathbb{P}_h, \tag{5.1.11}$$

Step 3: *Update* $s_h^{n+1} \in \mathbb{P}_h$ *such that*

$$\left\langle s_h^{n+1} \,,\, q_h \right\rangle = \left\langle s_h^n \,,\, q_h \right\rangle - \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \,,\, q_h \right\rangle, \quad \forall q_h \in \mathbb{P}_h, \tag{5.1.12}$$

Step 4: *Update* $\mathbf{u}_h^{n+1}$

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla \rho_h^{n+1}, \tag{5.1.13}$$

If necessary, compute $p_h^{n+1} \in \mathbb{P}_h$ *as follows:*

$$p_h^{n+1} = -\frac{\rho_h^{n+1}}{\triangle t} + \frac{1}{Re} s_h^{n+1}. \tag{5.1.14}$$

We note that $\mathbf{u}_h^{n+1}$ is discontinuous function across interelement boundaries, and that $\mathbf{u}_h^{n+1}$ is discrete divergence free in the sense that

$$\left\langle \mathbf{u}_h^{n+1}, \nabla \psi_h \right\rangle = 0, \quad \forall \psi_h \in P_h. \tag{5.1.15}$$

The gauge-Uzawa method not only preserves all advantages of gauge Algorithm 3.1 but also solves all difficulties regarding boundary calculation and space discretization. A chief advantage of Algorithm 5.2 is that there is no longer boundary and interior differentiation. This allows us to apply the Gauge-Uzawa easily to any domain in 2 and 3 dimension. In addition, this scheme is unconditionally stable and thus applicable to high Reynolds numbers, this is proved in the Section 5.2. The final results of this chapter are error estimates for velocity and also under the realistic regularity assumptions The proof of these theorems is several lemmas in Sections 5.3 and 5.4.

We know that the exact solution $(\mathbf{v}(t), p(t))$ for (1.1.1) is in $\mathbf{H}^2(\Omega) \times \mathbb{P}^1(\Omega)$ by Lemma 1.5 provided Assumptions 1-3 hold. In order to derive more generalized convergence results, we carry out the error analysis with the space regularity

$$(\mathbf{v}(t), p(t)) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega), \quad \text{where } \gamma \geq 1. \tag{5.1.16}$$

Therefore s is 1 in our main theorems. We define a constant κ as

$$\kappa = \min\{s, m\}, \tag{5.1.17}$$

where m is the polynomial degree of pressure p. Now, we state the main theorems of this chapter:

Theorem 5.1 *Let Assumptions 1-6 hold. If $h^2 \leq C\triangle t$, then we have*

$$\triangle t \sum_{n=0}^{N} \left\| \nabla \left(\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1} \right) \right\|_0^2 \leq C(\triangle t + h^{2\kappa}) \tag{5.1.18}$$

and

$$\triangle t \sum_{n=0}^{N} \left(\left\| \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1} \right\|_0^2 + \left\| \mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1} \right\|_0^2 \right) \leq C(\triangle t^2 + h^{2(\kappa+1)}). \tag{5.1.19}$$

And the error of pressure is

Theorem 5.2 *Let Assumptions 1-7 hold, and let $C_1 h^2 \leq \triangle t \leq C_2 h^{\frac{d}{2}}$ be valid with arbitrary constant C_1, $C_2 > 0$. If we have*

$$\left\| \nabla \mathbf{u}_t(0) \right\|_0 \leq M, \tag{5.1.20}$$

then we have

$$\triangle t \sum_{n=0}^{N} \left\| p(t_{n+1}) - p_h^{n+1} \right\|_0^2 \leq C \left(\triangle t + h^{2\kappa} \right). \tag{5.1.21}$$

5.2 Stability

In this section, we show that the Gauge-Uzawa method is unconditionally stable.

Lemma 5.1 *Let $s_h^{n+1} \in \mathbb{P}_h$ and $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ be defined in Algorithm 5.2. Then we have*

$$\left\| s_h^{n+1} - s_h^n \right\|_0 \leq \left\| \nabla \widehat{\mathbf{u}}_h^{n+1} \right\|_0. \tag{5.2.1}$$

Proof. Lemma 1.9 and formula (5.1.12) deduce

$$\begin{aligned} \left\| s_h^{n+1} - s_h^n \right\|_0^2 &= \left\langle s_h^{n+1} - s_h^n \,,\, s_h^{n+1} - s_h^n \right\rangle \\ &= -\left\langle s_h^{n+1} - s_h^n \,,\, \mathrm{div}\, \widehat{\mathbf{u}}_h^{n+1} \right\rangle \leq \left\| s_h^{n+1} - s_h^n \right\|_0 \left\| \nabla \widehat{\mathbf{u}}_h^{n+1} \right\|_0. \end{aligned} \tag{5.2.2}$$

So we have $\left\| s_h^{n+1} - s_h^n \right\|_0 \leq \left\| \nabla \widehat{\mathbf{u}}_h^{n+1} \right\|_0$, as asserted. ■

Theorem 5.3 (Stability) *The Gauge-Uzawa method is unconditionally stable in the sense that for all $\triangle t > 0$ the following priori bound holds:*

$$\begin{aligned} &\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n}\right\|_0^2 + \frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 + 2\sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 \\ &\quad + \frac{\triangle t}{Re}\left\|s_h^{N+1}\right\|_0^2 \le \left\|\mathbf{u}_h^0\right\|_0^2 + C\triangle t\sum_{n=0}^{N}\|\mathbf{f}(t_{n+1})\|_{-1}^2. \end{aligned} \tag{5.2.3}$$

Proof. By choosing $\mathbf{w}_h = 2\triangle t\widehat{\mathbf{u}}_h^{n+1}$ in the momentum equation (5.1.10) of Algorithm 5.2, we get

$$\begin{aligned} &2\left\langle\widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n}\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 2\triangle t\mathcal{N}(\mathbf{u}_h^{n}, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1}) \\ &+\frac{2\triangle t}{Re}\left\langle\nabla\widehat{\mathbf{u}}_h^{n+1}\,,\,\nabla\widehat{\mathbf{u}}_h^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\mathrm{div}\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle = 2\triangle t\left\langle\mathbf{f}(t_{n+1})\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle. \end{aligned} \tag{5.2.4}$$

We note that $\mathcal{N}(\mathbf{u}_h^{n}, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1})$ vanishes by (1.2.61). in view of (5.1.13) and (5.1.15), we have

$$\begin{aligned} 2\left\langle\widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n}\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle &= 2\left\langle\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n}\,,\,\mathbf{u}_h^{n+1}\right\rangle + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2 \\ &= \left\|\mathbf{u}_h^{n+1}\right\|_0^2 - \left\|\mathbf{u}_h^{n}\right\|_0^2 + \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n}\right\|_0^2 + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2, \end{aligned} \tag{5.2.5}$$

So the formula (5.2.4) becomes

$$\begin{aligned} &\left\|\mathbf{u}_h^{n+1}\right\|_0^2 - \left\|\mathbf{u}_h^{n}\right\|_0^2 + \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n}\right\|_0^2 + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ &\quad = \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\mathrm{div}\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 2\triangle t\left\langle\mathbf{f}(t_{n+1})\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle = A_1 + A_2. \end{aligned} \tag{5.2.6}$$

Then two right hand side terms can be bounded by Lemma 5.1 and (5.1.13), we derive

$$\begin{aligned} A_1 &= \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\mathrm{div}\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle \\ &= -\frac{2\triangle t}{Re}\left\langle s_h^n\,,\,s_h^{n+1} - s_h^n\right\rangle \\ &= -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2 - \left\|s_h^{n}\right\|_0^2 - \left\|s_h^{n+1} - s_h^{n}\right\|_0^2\right) \\ &\le -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2 - \left\|s_h^{n}\right\|_0^2\right) + \frac{\triangle t}{Re}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2, \end{aligned} \tag{5.2.7}$$

and

$$
\begin{aligned}
A_2 &= 2\triangle t \left\langle \mathbf{f}(t_{n+1}) \, , \, \widehat{\mathbf{u}}_h^{n+1} \right\rangle \\
&\le C\triangle t \|\mathbf{f}(t_{n+1})\|_{-1}^2 + \frac{\triangle t}{2Re} \left\| \nabla \widehat{\mathbf{u}}_h^{n+1} \right\|_0^2 .
\end{aligned}
\tag{5.2.8}
$$

Plugging (5.2.7) and (5.2.8) into (5.2.6) deduce

$$
\begin{aligned}
&\left\| \mathbf{u}_h^{n+1} \right\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \left\| \mathbf{u}_h^{n+1} - \mathbf{u}_h^n \right\|_0^2 + 2\left\| \nabla \rho_h^{n+1} \right\|_0^2 + \frac{\triangle t}{2Re} \left\| \nabla \widehat{\mathbf{u}}_h^{n+1} \right\|_0^2 \\
&\quad + \frac{\triangle t}{Re} \left(\left\| s_h^{n+1} \right\|_0^2 - \|s_h^n\|_0^2 \right) \le C\triangle t \|\mathbf{f}(t_{n+1})\|_{-1}^2 .
\end{aligned}
\tag{5.2.9}
$$

Summing over n from 0 to N implies (5.2.3). ■

5.3 Error Estimate for Velocity

In this section, we prove convergence of velocity the fully discrete Algorithm 5.2. To derive an error estimate under realistic regularity assumptions we have to deal with the fact that (1.2.12) does <u>not</u> imply $\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$. Instead, we handle the truncation error involving $\mathbf{u}_{tt}$ in the weaker space $L^2(\mathbf{Z})$ via Lemma 1.7, but at the expense of the additional difficulty that discrete functions are not in $\mathbf{Z}$, and so not divergence free. We thus introduce the auxiliary pair $(\mathbf{U}^{n+1}, P^{n+1}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, which is a weak solution of the following time discrete Stokes equations including exact convection:

$$
\left\{
\begin{aligned}
&\left\langle \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\triangle t} \, , \, \mathbf{w} \right\rangle + \frac{1}{Re} \left\langle \nabla \mathbf{U}^{n+1} \, , \, \nabla \mathbf{w} \right\rangle - \left\langle P^{n+1} \, , \, \operatorname{div} \mathbf{w} \right\rangle \\
&= \langle \mathbf{f}(t_{n+1}) \, , \, \mathbf{w} \rangle - \mathcal{N}\left(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w} \right), \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\
&\left\langle q \, , \, \operatorname{div} \mathbf{U}^{n+1} \right\rangle = 0, \qquad \forall q \in L^2(\Omega).
\end{aligned}
\right.
\tag{5.3.1}
$$

We use the following notation:

$$
\mathbf{G}^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{U}^{n+1} \quad \text{and} \quad g^{n+1} = p(t_{n+1}) - P^{n+1}. \tag{5.3.2}
$$

We proceed by first comparing (1.1.1) and (5.3.1), and estimating the truncation error in $L^2(0,T;\mathbf{Z}^*)$ because div $\mathbf{G}^{n+1}=0$. We next compare (5.3.1) with Algorithm 5.2.

Lemma 5.2 *Let Assumptions 1-3 hold. Then we have*

$$\left\|\mathbf{G}^{N+1}\right\|_0^2+\sum_{n=0}^{N}\left\|\mathbf{G}^{n+1}-\mathbf{G}^n\right\|_0^2+\frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\leq C\triangle t^2 \tag{5.3.3}$$

and

$$\triangle t\sum_{n=0}^{N}\left\|g^{n+1}\right\|_0^2\leq C\triangle t. \tag{5.3.4}$$

PROOF. The momentum equation (1.1.1) at time $t=t_{n+1}$ is

$$\mathbf{u}_t(t_{n+1})+(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})+\nabla p(t_{n+1})-\frac{1}{Re}\triangle\mathbf{u}(t_{n+1})=\mathbf{f}(t_{n+1}). \tag{5.3.5}$$

By virtue of the Taylor theorem,

$$\begin{aligned}\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\triangle t}&+(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})+\nabla p(t_{n+1})\\&-\frac{1}{Re}\triangle\mathbf{u}(t_{n+1})=\mathbf{R}_{n+1}+\mathbf{f}(t_{n+1}),\end{aligned} \tag{5.3.6}$$

where $\mathbf{R}_{n+1}=\dfrac{1}{\triangle t}\displaystyle\int_{t_n}^{t_{n+1}}(t-t_n)\mathbf{u}_{tt}(t)dt$ is the truncation error. Then (5.3.6) can be written in weak form as follows: for all $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$, we have

$$\begin{aligned}\left\langle\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\triangle t}\,,\,\mathbf{w}\right\rangle&+\frac{1}{Re}\left\langle\nabla\mathbf{u}(t_{n+1})\,,\,\nabla\mathbf{w}\right\rangle-\left\langle p(t_{n+1})\,,\,\mathrm{div}\,\mathbf{w}\right\rangle\\&=\left\langle\mathbf{R}_{n+1}\,,\,\mathbf{w}_h\right\rangle+\left\langle\mathbf{f}(t_{n+1})\,,\,\mathbf{w}\right\rangle-\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{w}).\end{aligned} \tag{5.3.7}$$

By subtracting (5.3.1) from (5.3.7), we can get

$$\begin{aligned}\left\langle\frac{\mathbf{G}^{n+1}-\mathbf{G}^n}{\triangle t}\,,\,\mathbf{w}\right\rangle&+\frac{1}{Re}\left\langle\nabla\mathbf{G}^{n+1}\,,\,\nabla\mathbf{w}\right\rangle\\&-\left\langle g^{n+1}\,,\,\mathrm{div}\,\mathbf{w}\right\rangle=\left\langle\mathbf{R}_{n+1}\,,\,\mathbf{w}\right\rangle.\end{aligned} \tag{5.3.8}$$

If we choose $\mathbf{w} = 2\triangle t \mathbf{G}^{n+1} \in \mathbf{H}_0^1(\Omega)$, then $\langle g^{n+1}, \operatorname{div} \mathbf{G}^{n+1} \rangle$ disappears because $\operatorname{div} \mathbf{G}^{n+1} = 0$. Then we have

$$\left\|\mathbf{G}^{n+1}\right\|_0^2 - \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}^{n+1} - \mathbf{G}^n\right\|_0^2 + \frac{\triangle t}{Re}\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 \le C\triangle t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2 dt.$$

By summation over n from 0 to N,

$$\left\|\mathbf{G}^{N+1}\right\|_0^2 + \sum_{n=0}^{N} \left\|\mathbf{G}^{n+1} - \mathbf{G}^n\right\|_0^2 + \frac{\triangle t}{Re} \sum_{n=0}^{N} \left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 \le C\triangle t^2 \int_{t_0}^{t_{N+1}} \|\mathbf{u}_{tt}(t)\|_{\mathbf{Z}^*}^2 dt.$$

In view of Lemma 1.7, we get (5.3.3). Using now (5.3.3), in conjunction with the continuous inf-sup condition, there exists $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that

$$\begin{aligned}
\|g^{n+1}\|_0^2 &= \left\langle g^{n+1}, \operatorname{div} \mathbf{w} \right\rangle \\
&= \left\langle \frac{\mathbf{G}^{n+1} - \mathbf{G}^n}{\triangle t}, \mathbf{w} \right\rangle + \frac{1}{Re} \left\langle \nabla \mathbf{G}^{n+1}, \nabla \mathbf{w} \right\rangle - \langle \mathbf{R}_{n+1}, \mathbf{w} \rangle \\
&\le C \left(\frac{\|\mathbf{G}^{n+1} - \mathbf{G}^n\|_0}{\triangle t} + \frac{1}{Re}\left\|\nabla \mathbf{G}^{n+1}\right\|_0 + \|\mathbf{R}_{n+1}\|_{-1} \right) \|\mathbf{w}\|_1
\end{aligned}$$

and

$$\|g^{n+1}\|_0 \le \frac{C}{\beta} \left(\frac{\|\mathbf{G}^{n+1} - \mathbf{G}^n\|_0}{\triangle t} + \frac{1}{Re}\left\|\nabla \mathbf{G}^{n+1}\right\|_0 + \|\mathbf{R}_{n+1}\|_{-1} \right). \tag{5.3.9}$$

Since

$$\begin{aligned}
\|\mathbf{R}_{n+1}\|_{-1}^2 &\le \int_{t_n}^{t_{n+1}} (t - t_n) \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt \\
&\le \int_{t_n}^{t_{n+1}} \sigma(t) \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt,
\end{aligned} \tag{5.3.10}$$

where $\sigma(t) = \min\{t, 1\}$, squaring the estimate for $\|g^{n+1}\|_0$ and multiplying $\triangle t$ yields

$$\begin{aligned}
\triangle t \left\|g^{n+1}\right\|_0^2 \le\ & \frac{C}{\beta^2} \frac{\|\mathbf{G}^{n+1} - \mathbf{G}^n\|_0^2}{\triangle t} + \frac{C\triangle t}{\beta^2 Re^2} \left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 \\
& + \frac{C\triangle t}{\beta^2} \int_{t_n}^{t_{n+1}} \sigma(t) \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt.
\end{aligned} \tag{5.3.11}$$

Adding over n from 0 to N, and using (1.2.12) and (5.3.3) leads to

$$\begin{aligned} \triangle t \sum_{n=0}^{N} \left\| g^{n+1} \right\|_0^2 &\leq \frac{C}{\triangle t} \sum_{n=0}^{N} \left\| \mathbf{G}^{n+1} - \mathbf{G}^n \right\|_0^2 + \frac{C\triangle t}{Re} \sum_{n=0}^{N} \left\| \nabla \mathbf{G}^{n+1} \right\|_0^2 \\ &\quad + C\triangle t \int_{t_0}^{t_{N+1}} \sigma(t) \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt \leq C\triangle t, \end{aligned} \tag{5.3.12}$$

which is the derived estimate (5.3.4). ■

Let $(\mathbf{U}_h^{n+1}, P_h^{n+1}) \in \mathbb{V}_h \times \mathbb{P}_h$ be a discrete solution of the following weak Stokes equations.

$$\begin{cases} \left\langle \nabla \mathbf{U}_h^{n+1}, \nabla \mathbf{w}_h \right\rangle + \left\langle P_h^{n+1}, \operatorname{div} \mathbf{w}_h \right\rangle \\ \qquad = \langle \nabla \mathbf{u}(t_{n+1}), \nabla \mathbf{w}_h \rangle + \langle \nabla p(t_{n+1}), \mathbf{w}_h \rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \left\langle r_h, \operatorname{div} \mathbf{U}_h^{n+1} \right\rangle = 0, \qquad \forall r_h \in \mathbb{P}_h, \end{cases} \tag{5.3.13}$$

where $(\mathbf{u}(t_{n+1}),\ p(t_{n+1}))$ is the exact solution of (1.1.1) at the time step t_{n+1}. Now, we define the new error functions

$$\begin{aligned} \mathbf{G}_h^{n+1} &= \mathbf{u}(t_{n+1}) - \mathbf{U}_h^{n+1}, \quad g_h^{n+1} = p(t_{n+1}) - P_h^{n+1}, \\ \mathbf{F}^{n+1} &= \mathbf{U}^{n+1} - \mathbf{U}_h^{n+1}, \qquad \eta^{n+1} = P^{n+1} - P_h^{n+1}. \end{aligned} \tag{5.3.14}$$

Then we have the following error estimate [12].

Lemma 5.3 *Let Assumptions 1-6 hold. And let the exact solution* $(\mathbf{u}(t_{n+1}), p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$. *Then we have*

$$\left\| \mathbf{G}_h^{n+1} \right\|_0 + h \left\| \mathbf{G}_h^{n+1} \right\|_1 \leq C h^{s+1} \left(\|\mathbf{u}(t_{n+1})\|_{s+1} + \|p(t_{n+1})\|_s \right). \tag{5.3.15}$$

Then we can derive the following Lemma easily,

Lemma 5.4 *Let Assumptions 1-6 hold. And let the exact solution* $(\mathbf{u}(t_{n+1}), p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$. *Then we have*

$$\left\| \left| \mathbf{G}_h^{n+1} \right| \right\| = \left\| \mathbf{G}_h^{n+1} \right\|_{\mathbf{L}^\infty(\Omega)} + \left\| \nabla \mathbf{G}_h^{n+1} \right\|_{\mathbf{L}^3(\Omega)} \leq M, \tag{5.3.16}$$

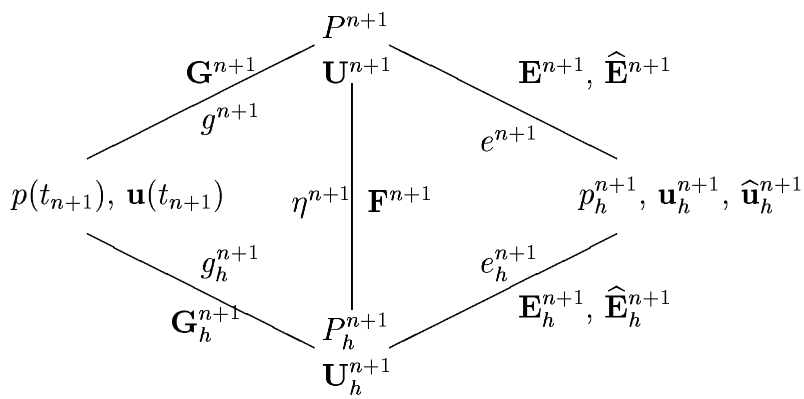


Table 5.1: The Notations of Error

$$\left\|g_h^{n+1}\right\|_0 \leq Ch^s \left(\|\mathbf{u}(t_{n+1})\|_{s+1} + \|p(t_{n+1})\|_s\right), \tag{5.3.17}$$

$$\left\|\mathbf{F}^{n+1}\right\|_0 \leq \left\|\mathbf{G}^{n+1}\right\|_0 + \left\|\mathbf{G}_h^{n+1}\right\|_0 \leq C(\triangle t + h^{s+1}), \tag{5.3.18}$$

$$\begin{aligned} \triangle t \sum_{n=0}^{N} \left\|\nabla \mathbf{F}^{n+1}\right\|_0^2 &\leq \triangle t \sum_{n=0}^{N} \left(\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla \mathbf{G}_h^{n+1}\right\|_0^2\right) \\ &\leq C(\triangle t^2 + h^{2s}), \end{aligned} \tag{5.3.19}$$

and

$$\triangle t \sum_{n=0}^{N} \left\|\eta^{n+1}\right\|_0^2 \leq \triangle t \sum_{n=0}^{N} \left(\left\|g^{n+1}\right\|_0^2 + \left\|g_h^{n+1}\right\|_0^2\right) \leq C(\triangle t + h^{2s}). \tag{5.3.20}$$

Now we use the following additional error functions:

$$\begin{aligned} &\mathbf{E}^{n+1} = \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}^{n+1} = \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e^{n+1} = P^{n+1} - p_h^{n+1}, \\ &\mathbf{E}_h^{n+1} = \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}_h^{n+1} = \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e_h^{n+1} = P_h^{n+1} - p_h^{n+1}, \end{aligned} \tag{5.3.21}$$

see Table 5.1 for the relation between (5.3.14) and (5.3.21). Then the properties of those error functions are:

Lemma 5.5 (Properties of Error Functions) *We have the following relations among the error functions in (5.3.21):*

$$\left\langle \mathbf{E}^{n+1}, \nabla q_h \right\rangle = \left\langle \mathbf{E}_h^{n+1}, \nabla q_h \right\rangle = \left\langle \mathbf{F}^{n+1}, \nabla q_h \right\rangle = 0, \quad \forall q_h \in \mathbb{P}_h, \tag{5.3.22}$$

$$
\begin{aligned}
\widehat{\mathbf{E}}^{n+1} &= \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1} \\
&= \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1} + \nabla \rho_h^{n+1} = \mathbf{E}^{n+1} + \nabla \rho_h^{n+1},
\end{aligned} \tag{5.3.23}
$$

$$
\begin{aligned}
\widehat{\mathbf{E}}_h^{n+1} &= \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1} \\
&= \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1} + \nabla \rho_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla \rho_h^{n+1},
\end{aligned} \tag{5.3.24}
$$

and

$$
\widehat{\mathbf{E}}^{n+1} = \mathbf{F}^{n+1} + \widehat{\mathbf{E}}_h^{n+1} \quad \textit{and} \quad \mathbf{E}^{n+1} = \mathbf{F}^{n+1} + \mathbf{E}_h^{n+1}. \tag{5.3.25}
$$

Lemma 5.6 *Let $s_h^{n+1} \in \mathbb{P}_h$ be defined in Algorithm 5.2. Then we have*

$$
\left\| s_h^{n+1} - s_h^n \right\|_0^2 \le \left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0^2. \tag{5.3.26}
$$

Proof. By Lemma 1.9 and by (5.1.12) in Algorithm 5.2,

$$
\begin{aligned}
\left\| s_h^{n+1} - s_h^n \right\|_0^2 &= \left\langle s_h^{n+1} - s_h^n \,,\, s_h^{n+1} - s_h^n \right\rangle \\
&= -\left\langle s_h^{n+1} - s_h^n \,,\, \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \right\rangle \\
&= \left\langle s_h^{n+1} - s_h^n \,,\, \operatorname{div} \widehat{\mathbf{E}}^{n+1} \right\rangle \\
&= \left\| s_h^{n+1} - s_h^n \right\|_0 \left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0. \qquad \blacksquare
\end{aligned} \tag{5.3.27}
$$

Our purpose in the following lemma is to show that both $\mathbf{u}_h^{n+1}$ and $\widehat{\mathbf{u}}_h^{n+1}$ are order $\mathcal{O}(\triangle t^{\frac{1}{2}} + h^s)$ approximations to $\mathbf{U}^{n+1}$ in $\mathbf{L}^2(\Omega)$. This result will be used to improve the error estimates to $\mathcal{O}(\triangle t + h^{s+1})$ orders in Lemma 5.10.

Lemma 5.7 *Let Assumptions 1-6 hold. And let the exact solution ($\mathbf{u}(t_{n+1})$, $p(t_{n+1})$) of (1.1.1) be in $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$, and let $h^2 \le C\triangle t$. Then we have*

$$
\begin{aligned}
&\left\| \mathbf{E}^{N+1} \right\|_0^2 + \left\| \widehat{\mathbf{E}}^{N+1} \right\|_0^2 + \frac{\triangle t}{Re} \left\| s_h^{N+1} \right\|_0^2 + \frac{1}{2} \sum_{n=0}^{N} \left\| \mathbf{E}^{n+1} - \mathbf{E}^n \right\|_0^2 \\
&+ \sum_{n=0}^{N} \left\| \nabla \rho_h^{n+1} \right\|_0^2 + \frac{\triangle t}{2Re} \sum_{n=0}^{N} \left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0^2 \le C(\triangle t + h^{2s}).
\end{aligned} \tag{5.3.28}
$$

PROOF. By subtracting (5.1.10) in Algorithm 5.2 from (5.3.1), we get , for all $\forall \mathbf{w}_h \in \mathbb{V}_h$,

$$\begin{aligned} &\left\langle \frac{\widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n}{\triangle t}, \mathbf{w}_h \right\rangle + \frac{1}{Re} \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{w}_h \right\rangle \\ &= \left\langle P^{n+1}, \operatorname{div} \mathbf{w}_h \right\rangle + \frac{1}{Re} \left\langle \nabla s_h^n, \mathbf{w}_h \right\rangle \\ &\quad - \mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w}_h) + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h). \end{aligned} \tag{5.3.29}$$

We choose $\mathbf{w}_h = 2\triangle t \widehat{\mathbf{E}}_h^{n+1} = 2\triangle t(\widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1})$, then (5.3.29) becomes

$$\begin{aligned} &2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \widehat{\mathbf{E}}_h^{n+1} \right\rangle + \frac{2\triangle t}{Re} \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \widehat{\mathbf{E}}_h^{n+1} \right\rangle \\ &= 2\triangle t \left\langle P^{n+1}, \operatorname{div} \widehat{\mathbf{E}}_h^{n+1} \right\rangle - \frac{2\triangle t}{Re} \left\langle s_h^n, \operatorname{div} \widehat{\mathbf{E}}_h^{n+1} \right\rangle \\ &\quad - 2\triangle t \mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1}) + 2\triangle t \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}). \end{aligned} \tag{5.3.30}$$

The left hand side of (5.3.30) becomes, by Lemma 5.5,

$$\begin{aligned} 2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \widehat{\mathbf{E}}_h^{n+1} \right\rangle &= 2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1} \right\rangle \\ &= \left\| \mathbf{E}^{n+1} \right\|_0^2 - \left\| \mathbf{E}^n \right\|_0^2 + \left\| \mathbf{E}^{n+1} - \mathbf{E}^n \right\|_0^2 \\ &\quad + 2\left\| \nabla \rho_h^{n+1} \right\|_0^2 - 2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n, \mathbf{F}^{n+1} \right\rangle \end{aligned} \tag{5.3.31}$$

and

$$\frac{2\triangle t}{Re} \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \widehat{\mathbf{E}}_h^{n+1} \right\rangle = \frac{2\triangle t}{Re} \left(\left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0^2 - \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{F}^{n+1} \right\rangle \right). \tag{5.3.32}$$

By (5.3.31) and (5.3.32), the formula (5.3.30) can be split by

$$\begin{aligned}
&\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2 \\
&= 2\left\langle\mathbf{E}^{n+1} - \mathbf{E}^n\,,\ \mathbf{F}^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle\nabla\widehat{\mathbf{E}}^{n+1}\,,\ \nabla\mathbf{F}^{n+1}\right\rangle \\
&\quad +2\triangle t\left\langle P^{n+1}\,,\ \mathrm{div}\ \widehat{\mathbf{E}}_h^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle s_h^n\,,\ \mathrm{div}\ \widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1}) - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1})\right) \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned} \tag{5.3.33}$$

By Hölder inequality, the first two terms can be treated as

$$A_1 \le \frac{1}{2}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + C\left\|\mathbf{F}^{n+1}\right\|_0^2 \tag{5.3.34}$$

and

$$A_2 \le \frac{\triangle t}{8Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\triangle t}{Re}\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2. \tag{5.3.35}$$

By Lemmas 1.9 and 5.5, we derive

$$\begin{aligned}
A_3 &= 2\triangle t\left\langle\eta^{n+1}\,,\ \mathrm{div}\ \widehat{\mathbf{E}}_h^{n+1}\right\rangle + 2\triangle t\left\langle P_h^{n+1}\,,\ \mathrm{div}\ \widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&= 2\triangle t\left\langle\eta^{n+1}\,,\ \mathrm{div}\ \widehat{\mathbf{E}}_h^{n+1}\right\rangle - 2\triangle t\left\langle\nabla P_h^{n+1}\,,\ \nabla\rho_h^{n+1}\right\rangle \\
&\le CRe\triangle t\left\|\eta^{n+1}\right\|_0^2 + \frac{\triangle t}{8Re}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{F}^{n+1}\right\|_0^2\right) \\
&\quad +C\triangle t^2\left(\|\mathbf{u}(t_{n+1})\|_2^2 + \|\nabla p(t_{n+1})\|_0^2\right) + \left\|\nabla\rho_h^{n+1}\right\|_0^2.
\end{aligned} \tag{5.3.36}$$

The above A_1, A_2, and A_3 are the bad terms to get optimal order $\mathcal{O}(\triangle t) + \mathcal{O}(h^{s+1})$. By the formula (5.1.12) and Lemmas 5.5-5.6, we obtain

$$\begin{aligned} A_4 &= \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\mathrm{div}\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle \\ &= -\frac{2\triangle t}{Re}\left\langle s_h^n\,,\,s_h^{n+1}-s_h^n\right\rangle \\ &= -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2 - \|s_h^n\|_0^2 - \left\|s_h^{n+1}-s_h^n\right\|_0^2\right) \\ &\le -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2 - \|s_h^n\|_0^2\right) + \frac{\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \end{aligned} \tag{5.3.37}$$

In the estimation of convection A_5, we will use frequently Lemmas We split the remaining term A_5, the only one dealing with convection, as follows: 1.5 and 1.16.

$$\begin{aligned} A_5 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}_h^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}_h^n,\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}_h^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}_h^n,\mathbf{G}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}) - 2\triangle t\mathcal{N}(\mathbf{u}_h^n,\widehat{\mathbf{E}}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}) \\ &= = A_{5,1}+A_{5,2}+A_{5,3}+A_{5,4}. \end{aligned} \tag{5.3.38}$$

We note $A_{5,4}$ vanishes by (1.2.61). Since $\|\mathbf{u}(t_{n+1})\|_2 \le C$, the first two terms can be written by

$$\begin{aligned} A_{5,1} &\le C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\ &\le CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt + \frac{\triangle t}{24Re}\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2, \end{aligned} \tag{5.3.39}$$

and

$$\begin{aligned} A_{5,2} &\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\ &\le CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2\right) + \frac{\triangle t}{24Re}\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2. \end{aligned} \tag{5.3.40}$$

Since we have $\left\|\mathbf{G}_h^{n+1}\right\| \leq M$ by 5.4, we derive

$$
\begin{aligned}
A_{5,3} &= -2\triangle t\mathcal{N}(\mathbf{u}_h^n - \mathbf{u}(t_n), \mathbf{G}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n), \mathbf{G}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}) \\
&\leq C\triangle t\|\mathbf{u}_h^n - \mathbf{u}(t_n)\|_0 \left\|\mathbf{G}_h^{n+1}\right\| \left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\
&\quad +C\triangle t\|\mathbf{u}(t_n)\|_2 \left\|\mathbf{G}_h^{n+1}\right\|_0 \left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\
&\leq CRe\triangle t\left(\|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2\right) + \frac{\triangle t}{24Re}\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2.
\end{aligned}
\tag{5.3.41}
$$

Since $\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 \leq 2\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + 2\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2$, plugging (5.3.39)-(5.3.41) into (5.3.38) deduce

$$
\begin{aligned}
A_5 &\leq CRe\triangle t\left(\|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2\right) + \frac{\triangle t}{4Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\
&\quad +\frac{\triangle t}{4Re}\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2 + CRe\triangle t^2\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned}
\tag{5.3.42}
$$

By Replacing (5.3.34)-(5.3.37) and (5.3.42) on the formula (5.3.33), We derive

$$
\begin{aligned}
&\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \frac{1}{2}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla\rho_h^{n+1}\right\|_0^2 \\
&+\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2 - \|s_h^n\|_0^2\right) \leq C\left\|\mathbf{F}^{n+1}\right\|_0^2 + \frac{C\triangle t}{Re}\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2 \\
&\quad +CRe\triangle t\left(\|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2 + \left\|\eta^{n+1}\right\|_0^2\right) \\
&\quad +C\triangle t^2\left(\|\mathbf{u}(t_{n+1})\|_2^2 + \|\nabla p(t_{n+1})\|_0^2\right) + CRe\triangle t^2\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned}
\tag{5.3.43}
$$

Summing over n from 0 to N,

$$\begin{aligned}
&\|\mathbf{E}^{N+1}\|_0^2 + \frac{\triangle t}{Re}\|s_h^{N+1}\|_0^2 + \frac{1}{2}\sum_{n=0}^{N}\|\mathbf{E}^{n+1} - \mathbf{E}^n\|_0^2 + \sum_{n=0}^{N}\|\nabla\rho_h^{n+1}\|_0^2 \\
&+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \le \|\mathbf{E}^0\|_0^2 + \frac{C\triangle t}{Re}\sum_{n=0}^{N}\|\nabla\mathbf{F}^{n+1}\|_0^2 \\
&+C\sum_{n=0}^{N}\|\mathbf{F}^{n+1}\|_0^2 + C\triangle t^2\sum_{n=0}^{N}\left(\|\mathbf{u}(t_{n+1})\|_2^2 + \|\nabla p(t_{n+1})\|_0^2\right) \\
&+CRe\triangle t\sum_{n=0}^{N}\left(\|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2 + \left\|\eta^{n+1}\right\|_0^2\right) \\
&+CRe\triangle t^2\int_{t_0}^{t_{N+1}}\|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned}$$

We note the assumption $h^2 \le C\triangle t$ need to estimate $\sum_{n=0}^{N}\|\mathbf{F}^{n+1}\|_0^2$. Since $\|\mathbf{E}^0\|_0^2 \le Ch^{2s+2}\|\mathbf{u}(0)\|_{s+1}^2$, by discrete Gronwall lemma, and by Lemmas 5.2 and 5.4,

$$\begin{aligned}
&\|\mathbf{E}^{N+1}\|_0^2 + \frac{\triangle t}{Re}\|s_h^{N+1}\|_0^2 + \frac{1}{2}\sum_{n=0}^{N}\|\mathbf{E}^{n+1} - \mathbf{E}^n\|_0^2 + \sum_{n=0}^{N}\|\nabla\rho_h^{n+1}\|_0^2 \\
&\qquad+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \le C(\triangle t + h^{2s}).
\end{aligned} \tag{5.3.44}$$

By (5.3.23) in Lemma 5.5, we have $\left\|\widehat{\mathbf{E}}^{N+1}\right\|_0^2 = \|\mathbf{E}^{N+1}\|_0^2 + \left\|\nabla\rho_h^{N+1}\right\|_0^2$. So we get (5.3.28). ■

Let $(\mathbf{v}^{n+1}, q^{n+1}) \in \mathbf{H}^{s+1}(\Omega)\times H^s(\Omega)$ be the weak solution of the following Stokes equations:

$$\begin{cases}
-\triangle\mathbf{v}^{n+1} + \nabla q^{n+1} = \mathbf{E}^{n+1}, & \text{in } \Omega, \\
\text{div }\mathbf{v}^{n+1} = 0, & \text{in } \Omega, \\
\mathbf{v}^{n+1} = 0, & \text{on } \partial\Omega.
\end{cases} \tag{5.3.45}$$

Let $(\mathbf{v}_h^{n+1}, q_h^{n+1}) \in \mathbb{V}_h \times \mathbb{P}_h$ be a discrete solution of

$$\begin{cases} \langle \nabla \mathbf{v}_h^{n+1}, \nabla \mathbf{w}_h \rangle - \langle q_h^{n+1}, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{E}^{n+1}, \mathbf{w}_h \rangle, & \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle \nabla r_h, \mathbf{v}_h^{n+1} \rangle = 0, & \forall r_h \in \mathbb{P}_h. \end{cases} \tag{5.3.46}$$

The following error estimate follows from [12] and Lemma 1.17.

Lemma 5.8 *(Properties of $\mathbf{v}^{n+1}$ and $\mathbf{v}_h^{n+1}$) Let Assumptions 1-6 hold. We have*

$$\begin{aligned} \|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\|_0 + h\|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\|_1 &\leq Ch^2 \left(\|\mathbf{v}^{n+1}\|_2 + \|p^{n+1}\|_1\right) \\ &\leq Ch^2 \|\mathbf{E}^{n+1}\|_0, \end{aligned} \tag{5.3.47}$$

$$|||\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}||| \leq C\|\mathbf{E}^{n+1}\|_0, \tag{5.3.48}$$

and

$$\mathbf{v}_h^0 = 0. \tag{5.3.49}$$

Lemma 5.9 *Let $\mathbf{v}^{n+1}$ and $\mathbf{v}_h^{n+1}$ be the solutions of (5.3.45) and (5.3.46), respectively. Then we have*

$$\|\mathbf{E}^{n+1}\|_{\mathbf{Z}^*} \leq Ch\|\mathbf{E}^{n+1}\|_0 + C\|\nabla \mathbf{v}_h^{n+1}\|_0. \tag{5.3.50}$$

Proof. From Lemma 1.1, we obtain

$$\begin{aligned} \|\mathbf{E}^{n+1}\|_{\mathbf{Z}^*} &\leq C\|\nabla \mathbf{v}^{n+1}\|_0 \\ &\leq C\|\nabla(\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1})\|_0 + C\|\nabla \mathbf{v}_h^{n+1}\|_0 \\ &\leq Ch\|\mathbf{E}^{n+1}\|_0 + C\|\nabla \mathbf{v}_h^{n+1}\|_0. \qquad \blacksquare \end{aligned} \tag{5.3.51}$$

Lemma 5.10 *Let Assumptions 1-6 hold. And let the exact solution* $(\mathbf{u}(t_{n+1}),$ $p(t_{n+1}))$ *of (1.1.1) be in* $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$*, and let* $h^2 \leq C\triangle t$*. Then we have*

$$\begin{aligned} &\left\|\mathbf{E}^{N+1}\right\|_{\mathbf{Z}^*}^2 + \sum_{n=0}^{N} \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_{\mathbf{Z}^*}^2 + \frac{\triangle t}{Re} \sum_{n=0}^{N} \left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \right) \\ &\qquad \leq C\left(\triangle t^2 + h^{2s+2}\right). \end{aligned} \tag{5.3.52}$$

Proof. We choose $\mathbf{w}_h = 2\triangle t \mathbf{v}_h^{n+1}$ in formula (5.3.29), then we have

$$\begin{aligned} &2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \mathbf{v}_h^{n+1} \right\rangle + \frac{2\triangle t}{Re} \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{v}_h^{n+1} \right\rangle \\ &= 2\triangle t \left\langle P^{n+1}, \operatorname{div} \mathbf{v}_h^{n+1} \right\rangle - \frac{2\triangle t}{Re} \left\langle s_h^n, \operatorname{div} \mathbf{v}_h^{n+1} \right\rangle \\ &\quad -2\triangle t \mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h^{n+1}) + 2\triangle t \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1}). \end{aligned} \tag{5.3.53}$$

We note $\left\langle \nabla s_h, \mathbf{v}_h^{n+1} \right\rangle = 0$. The terms of left hand side in (5.3.53) become, by the weak formulation (5.3.46),

$$\begin{aligned} 2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \mathbf{v}_h^{n+1} \right\rangle &= 2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n, \mathbf{v}_h^{n+1} \right\rangle \\ &= 2\left\langle \nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n), \nabla \mathbf{v}_h^{n+1} \right\rangle \\ &= \left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2 - \left\|\nabla \mathbf{v}_h^n\right\|_0^2 + \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 \end{aligned} \tag{5.3.54}$$

and

$$\begin{aligned} \frac{2\triangle t}{Re} \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{v}_h^{n+1} \right\rangle &= \frac{2\triangle t}{Re} \left\langle \nabla(\mathbf{F}^{n+1} + \widehat{\mathbf{E}}_h^{n+1}), \nabla \mathbf{v}_h^{n+1} \right\rangle \\ &= \frac{2\triangle t}{Re} \left(\left\langle \nabla \mathbf{F}^{n+1}, \nabla \mathbf{v}_h^{n+1} \right\rangle + \left\langle \widehat{\mathbf{E}}_h^{n+1}, \mathbf{E}^{n+1} - \nabla q_h^{n+1} \right\rangle \right) \\ &= \frac{2\triangle t}{Re} \left(\left\langle \nabla \mathbf{F}^{n+1}, \nabla \mathbf{v}_h^{n+1} \right\rangle + \left\|\mathbf{E}^{n+1}\right\|_0^2 \right. \\ &\quad \left. - \left\langle \mathbf{F}^{n+1}, \mathbf{E}^{n+1} \right\rangle - \left\langle \nabla \rho_h^{n+1}, \nabla q_h^{n+1} \right\rangle \right). \end{aligned} \tag{5.3.55}$$

From (5.3.54) and (5.3.55), formula (5.3.53) can be rewritten by

$$\begin{aligned}
&\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2 - \|\nabla \mathbf{v}_h^n\|_0^2 + \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
&= -\frac{2\triangle t}{Re}\left\langle \nabla \mathbf{F}^{n+1}, \nabla \mathbf{v}_h^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle \mathbf{F}^{n+1}, \mathbf{E}^{n+1}\right\rangle \\
&\quad + \frac{2\triangle t}{Re}\left\langle \nabla \rho_h^{n+1}, \nabla q_h^{n+1}\right\rangle + 2\triangle t\left\langle P^{n+1}, \operatorname{div} \mathbf{v}_h^{n+1}\right\rangle \qquad (5.3.56)\\
&\quad - 2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h^{n+1}) - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1})\right) \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

Assumption 1 and Lemma 5.8 help us to estimate the first term in (5.3.56).

$$\begin{aligned}
A_1 &= \frac{2\triangle t}{Re}\left\langle \nabla \mathbf{F}^{n+1}, \nabla\left(\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right)\right\rangle - \frac{2\triangle t}{Re}\left\langle \nabla \mathbf{F}^{n+1}, \nabla \mathbf{v}^{n+1}\right\rangle \\
&\le \frac{C\triangle th}{Re}\left\|\nabla \mathbf{F}^{n+1}\right\|_0\left\|\mathbf{E}^{n+1}\right\|_0 + \frac{C\triangle t}{Re}\left\|\mathbf{F}^{n+1}\right\|_0\left\|\mathbf{v}^{n+1}\right\|_2 \qquad (5.3.57)\\
&\le \frac{C\triangle t}{Re}\left(h^2\left\|\nabla \mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{F}^{n+1}\right\|_0^2\right) + \frac{\triangle t}{8Re}\left\|\mathbf{E}^{n+1}\right\|_0^2.
\end{aligned}$$

The following two terms can be deduced easily by

$$\begin{aligned}
A_2 &\le \frac{C\triangle t}{Re}\left\|\mathbf{F}^{n+1}\right\|_0\left\|\mathbf{E}^{n+1}\right\|_0 \\
&\le \frac{C\triangle t}{Re}\left\|\mathbf{F}^{n+1}\right\|_0^2 + \frac{\triangle t}{8Re}\left\|\mathbf{E}^{n+1}\right\|_0^2
\end{aligned} \qquad (5.3.58)$$

and

$$A_3 \le \frac{C\triangle t}{Re}\left\|\nabla \rho_h^{n+1}\right\|_0^2 + \frac{\triangle t}{8Re}\left\|\mathbf{E}^{n+1}\right\|_0^2. \qquad (5.3.59)$$

By incompressible constraint in discrete Stokes equations (5.3.46), we get

$$\begin{aligned}
A_4 &= 2\triangle t\left\langle P^{n+1} - P_h^{n+1}, \operatorname{div}\left(\mathbf{v}_h^{n+1} - \mathbf{v}^{n+1}\right)\right\rangle \\
&\le C\triangle th\left\|\eta^{n+1}\right\|_0\left\|\mathbf{v}^{n+1}\right\|_2 \qquad (5.3.60)\\
&\le CRe\triangle th^2\left\|\eta^{n+1}\right\|_0^2 + \frac{\triangle t}{8Re}\left\|\mathbf{E}^{n+1}\right\|_0^2.
\end{aligned}$$

Now we carry out the convection A_5 term by estimating separately each split term.

$$
\begin{aligned}
A_5 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}_h^n,\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}_h^n,\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}) = A_{5,1}+A_{5,2}+A_{5,3}.
\end{aligned} \tag{5.3.61}
$$

The two Lemmas 1.5 and 1.16 give us the following estimations

$$
\begin{aligned}
A_{5,1} &\leq C\triangle t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0 \\
&\leq C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt + C\triangle t\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2,
\end{aligned} \tag{5.3.62}
$$

$$
\begin{aligned}
A_{5,2} &\leq C\triangle t\left(\|\mathbf{E}^n\|_0+\|\mathbf{G}^n\|_0\right)\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0 \\
&\leq \frac{\triangle t}{8Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\|\mathbf{G}^n\|_0^2\right) \\
&\quad +CRe\triangle t\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2,
\end{aligned} \tag{5.3.63}
$$

and

$$
\begin{aligned}
A_{5,3} &= 2\triangle t\mathcal{N}\left(\mathbf{u}(t_n)-\mathbf{u}_h^n,\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}\right) \\
&\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_n),\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}\right) \\
&= 2\triangle t\mathcal{N}\left(\mathbf{u}(t_n)-\mathbf{u}_h^n,\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}^{n+1}\right) \\
&\quad +2\triangle t\mathcal{N}\left(\mathbf{u}(t_n)-\mathbf{u}_h^n,\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}^{n+1}\right) \\
&\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_n),\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}\right) = A_{5,4}+A_{5,5}+A_{5,6}.
\end{aligned} \tag{5.3.64}
$$

By Lemmas 5.7 and 5.8, the three terms on the right-hand side of (5.3.64) can be estimated as follows:

$$\begin{aligned} A_{5,4} &\le C\triangle t\|\mathbf{G}^n+\mathbf{E}^n\|_0\left\|\nabla(\mathbf{G}^{n+1}+\widehat{\mathbf{E}}^{n+1})\right\|_0|\!|\!|\mathbf{v}_h^{n+1}-\mathbf{v}^{n+1}|\!|\!| \\ &\le C\triangle t\left(\triangle t^{\frac{1}{2}}+h^s\right)\left(\|\nabla\mathbf{G}^{n+1}\|_0+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\right)\|\mathbf{E}^{n+1}\|_0 \\ &\le CRe\triangle t\left(\triangle t+h^{2s}\right)\left(\|\nabla\mathbf{G}^{n+1}\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\ &\quad +\frac{\triangle t}{8Re}\|\mathbf{E}^{n+1}\|_0^2, \end{aligned} \tag{5.3.65}$$

$$\begin{aligned} A_{5,5} &\le C\triangle t\|\mathbf{G}^n+\mathbf{E}^n\|_0\left\|\nabla(\mathbf{G}^{n+1}+\widehat{\mathbf{E}}^{n+1})\right\|_0\|\mathbf{v}^{n+1}\|_2 \\ &\le C\triangle t\left(\triangle t^{\frac{1}{2}}+h^s\right)\left(\|\nabla\mathbf{G}^{n+1}\|_0+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\right)\|\mathbf{E}^{n+1}\|_0 \\ &\le CRe\triangle t\left(\triangle t+h^{2s}\right)\left(\|\nabla\mathbf{G}^{n+1}\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\ &\quad +\frac{\triangle t}{8Re}\|\mathbf{E}^{n+1}\|_0^2, \end{aligned} \tag{5.3.66}$$

and

$$\begin{aligned} A_{5,6} &\le C\triangle t\|\mathbf{u}(t_n)\|_2\left\|\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1}\right\|_0\left\|\mathbf{v}_h^{n+1}\right\|_1 \\ &\le C\triangle t\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0+\|\mathbf{G}^{n+1}\|_0\right)\|\nabla\mathbf{v}_h^{n+1}\|_0 \\ &\le \frac{\triangle t}{8Re}\left(\|\mathbf{E}^{n+1}\|_0^2+\|\nabla\rho_h^{n+1}\|_0^2+\|\mathbf{G}^{n+1}\|_0^2\right) \\ &\quad +CRe\triangle t\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2. \end{aligned} \tag{5.3.67}$$

Upon Plugging (5.3.62)-(5.3.67) into (5.3.61), we obtain

$$\begin{aligned} A_5 \le\ & \frac{\triangle t}{2Re}\left(\|\mathbf{E}^{n+1}\|_0^2+\|\mathbf{E}^{n+1}-\mathbf{E}^n\|_0^2+\|\mathbf{G}^{n+1}\|_0^2+\|\mathbf{G}^n\|_0^2\right) \\ &+CRe\triangle t(\triangle t+h^{2s})\left(\|\nabla\mathbf{G}^{n+1}\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\ &+\frac{\triangle t}{8Re}\|\nabla\rho_h^{n+1}\|_0^2+CRe\triangle t\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 \\ &+C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt. \end{aligned} \tag{5.3.68}$$

Gathering (5.3.56)-(5.3.60) and (5.3.68), we obtain

$$
\begin{aligned}
&\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2 - \|\nabla \mathbf{v}_h^n\|_0^2 + \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 + \frac{\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
&\leq CRe\triangle t\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2 + CRe\triangle t h^2\left\|\eta^{n+1}\right\|_0^2 + C\triangle t^2\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_0^2 dt \\
&\quad + \frac{C\triangle t}{Re}\left(h^2\left\|\nabla \mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2\right) \\
&\quad + \frac{C\triangle t}{Re}\left(\left\|\mathbf{G}^{n+1}\right\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\nabla \rho_h^{n+1}\right\|_0^2\right) \\
&\quad + CRe\triangle t(\triangle t + h^{2s})\left(\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2\right).
\end{aligned} \tag{5.3.69}
$$

On adding over n from 0 to N,

$$
\begin{aligned}
&\left\|\nabla \mathbf{v}_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
&\leq CRe\triangle t\sum_{n=0}^{N}\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2 + CRe\triangle t h^2\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|_0^2 \\
&\quad + \frac{C\triangle t}{Re}\sum_{n=0}^{N}\left(h^2\left\|\nabla \mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2\right) \\
&\quad + \frac{C\triangle t}{Re}\sum_{n=0}^{N+1}\left(\|\mathbf{G}^n\|_0^2 + \|\nabla \rho_h^n\|_0^2\right) + C\triangle t^2\int_{t_0}^{t_{N+1}} \|\mathbf{u}_t(t)\|_0^2 dt \\
&\quad + CRe\triangle t(\triangle t + h^{2s})\sum_{n=0}^{N}\left(\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2\right).
\end{aligned} \tag{5.3.70}
$$

Since Lemma 5.7 support

$$
\begin{aligned}
&\sum_{n=0}^{N}\left(\left\|\nabla \rho_h^{n+1}\right\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2\right) + \triangle t\sum_{n=0}^{N}\left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\
&\qquad \leq C\left(\triangle t + h^{2s}\right),
\end{aligned} \tag{5.3.71}
$$

the formula (5.3.70) can be simplified by Lemmas 5.2 and 5.4,

$$\begin{aligned}&\left\|\nabla \mathbf{v}_h^{N+1}\right\|_0^2+\sum_{n=0}^{N}\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+\frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2\\&\leq C\left(\triangle t^2+h^{2s+2}+\triangle t h^{2s}\right)+C\triangle t\sum_{n=0}^{N}\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2.\end{aligned} \tag{5.3.72}$$

By discrete Gronwall lemma and Lemma 5.9, we derive

$$\begin{aligned}&\left\|\mathbf{E}^{N+1}\right\|_{\mathbf{Z}^*}^2+\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_{\mathbf{Z}^*}^2+\frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2\\&\leq C\left(\triangle t^2+h^{2s+2}\right).\end{aligned} \tag{5.3.73}$$

From Lemmas 5.5 and 5.7, we drive

$$\begin{aligned}\triangle t\sum_{n=0}^{N}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 &= \triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla \rho_h^{n+1}\right\|_0^2\right)\\&\leq C\left(\triangle t^2+h^{2s+2}\right).\end{aligned} \tag{5.3.74}$$

So we proved (5.3.52). ■

5.4 Error Estimate for Pressure

The goal of this section is to estimate the pressure error in $L^2(L^2)$ for Algorithm 5.2. The main difficulty is to derive the following improved error estimate:

$$\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 \leq C(\triangle t^2+h^{2s+2}), \tag{5.4.1}$$

Since we just proved a suboptimal order $\mathcal{O}(\sqrt{\triangle t}+h^s)$ in Theorem 5.7, showing (5.4.1) requires more regularity of the exact solution of (5.4.2).

Lemma 5.11 *Let Assumptions 1-3 hold and*

$$\int_0^T \|\mathbf{u}_{tt}\|_0^2 dt \leq M. \tag{5.4.2}$$

Then we have

$$\frac{1}{2}\left\|\mathbf{G}^{N+1}-\mathbf{G}^{N}\right\|_0^2 + \frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\|_0^2 + \frac{2\triangle t}{Re}\sum_{n=1}^{N}\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\right\|_0^2 \le M\triangle t^3, \tag{5.4.3}$$

$$\max_{0\le n\le N+1}\|g^n\|_0^2 \le C\triangle t, \tag{5.4.4}$$

and

$$\triangle t\sum_{n=0}^{N}\left\|g^{n+1}-g^{n}\right\|_0^2 \le C\triangle t^2. \tag{5.4.5}$$

PROOF. Subtracting n time step from $n+1$ time step for (5.3.8) yields

$$\left\langle \frac{\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}}{\triangle t}\,,\,\mathbf{w}\right\rangle + \frac{1}{Re}\left\langle \nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\,,\,\nabla\mathbf{w}\right\rangle - \left\langle g^{n+1}-g^{n}\,,\,\mathrm{div}\,\mathbf{w}\right\rangle = \left\langle \mathbf{R}_{n+1}-\mathbf{R}_{n}\,,\,\mathbf{w}\right\rangle. \tag{5.4.6}$$

If we choose $\mathbf{w} = 2\triangle t(\mathbf{G}^{n+1}-\mathbf{G}^{n}) \in \mathbf{H}_0^1(\Omega)$, then $\langle g^{n+1}-g^{n}\,,\,\mathrm{div}\,\mathbf{G}^{n+1}-\mathbf{G}^{n}\rangle$ vanishes because $\mathrm{div}\,\mathbf{G}^{n+1}=0$. Then we have

$$\left\|\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\|_0^2 - \left\|\mathbf{G}^{n}-\mathbf{G}^{n-1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\right\|_0^2 = 2\triangle t\left\langle \mathbf{R}_{n+1}-\mathbf{R}_{n}\,,\,\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\rangle. \tag{5.4.7}$$

Since the right hand side of (5.4.7) becomes

$$\begin{aligned}
&\left\langle \mathbf{R}_{n+1}\,,\,\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\rangle - \left\langle \mathbf{R}_{n}\,,\,\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\rangle \\
&= \left\langle \mathbf{R}_{n+1}\,,\,\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\rangle - \left\langle \mathbf{R}_{n}\,,\,\mathbf{G}^{n}-\mathbf{G}^{n-1}\right\rangle \\
&\quad - \left\langle \mathbf{R}_{n}\,,\,\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\rangle,
\end{aligned} \tag{5.4.8}$$

summation over n from 1 to N yield

$$
\begin{aligned}
&\left\|\mathbf{G}^{N+1}-\mathbf{G}^{N}\right\|_0^2+\sum_{n=1}^{N}\left\|\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\|_0^2\\
&+\frac{2\triangle t}{Re}\sum_{n=1}^{N}\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\right\|_0^2\\
&=2\triangle t\left\langle \mathbf{R}_{N+1}\,,\,\mathbf{G}^{N+1}-\mathbf{G}^{N}\right\rangle-2\triangle t\left\langle \mathbb{R}_1\,,\,\mathbf{G}^{1}-\mathbf{G}^{0}\right\rangle \qquad (5.4.9)\\
&\quad -2\triangle t\sum_{n=1}^{N}\left\langle \mathbf{R}_n\,,\,\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\rangle+\left\|\mathbf{G}^{1}-\mathbf{G}^{0}\right\|_0^2\\
&=A_1+A_2+A_3+A_4.
\end{aligned}
$$

We now estimate splinted terms in (5.4.9) separated as follows:

$$A_1\le C\triangle t^3\int_{t_N}^{t_{N+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt+\frac{1}{2}\left\|\mathbf{G}^{N+1}-\mathbf{G}^{N}\right\|_0^2, \qquad (5.4.10)$$

$$A_2\le C\triangle t^3\int_{t_0}^{t_1}\|\mathbf{u}_{tt}(t)\|_0^2dt+\left\|\mathbf{G}^{1}-\mathbf{G}^{0}\right\|_0^2, \qquad (5.4.11)$$

and

$$A_3\le C\triangle t^3\int_{t_0}^{t_{N+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt+\frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\|_0^2. \qquad (5.4.12)$$

Upon plugging (5.4.10)-(5.4.12) into (5.4.9) we derive

$$
\begin{aligned}
&\frac{2\triangle t}{Re}\sum_{n=1}^{N}\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\right\|_0^2+\frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{G}^{n+1}-2\mathbf{G}^{n}+\mathbf{G}^{n-1}\right\|_0^2\\
&+\frac{1}{2}\left\|\mathbf{G}^{N+1}-\mathbf{G}^{N}\right\|_0^2\le C\triangle t^3\int_{t_0}^{t_{N+1}}\|\mathbf{u}_{tt}\|_0^2dt+2\left\|\mathbf{G}^{1}-\mathbf{G}^{0}\right\|_0^2.
\end{aligned} \qquad (5.4.13)
$$

In order to prove $\|\mathbf{G}^1-\mathbf{G}^0\|_0^2\le C\triangle t^3$, which is the sufficient condition to get (5.4.3), we choose $n=0$ and $\mathbf{w}=2\triangle t\mathbf{G}^1$ in (5.3.8). Then we get

$$
\begin{aligned}
&\left\|\mathbf{G}^{1}\right\|_0^2-\left\|\mathbf{G}^{0}\right\|_0^2+\left\|\mathbf{G}^{1}-\mathbf{G}^{0}\right\|_0^2+\frac{\triangle t}{Re}\left\|\nabla\mathbf{G}^{1}\right\|_0^2\\
&\le C\triangle t^3\int_{t_0}^{t_1}\|\mathbf{u}_{tt}(t)\|_0^2dt+\left\|\mathbf{G}^{1}\right\|_0^2.
\end{aligned} \qquad (5.4.14)
$$

Since $\|\mathbf{G}^0\|_0^2 = 0$, we derive $\|\mathbf{G}^1 - \mathbf{G}^0\|_0^2 \le C\triangle t^3$. Therefore we get (5.4.3). Also we get (5.4.4) by considering (5.3.11) and (5.4.2)-(5.4.3). Now we prove (5.4.5) by using (5.4.3) and (5.4.6). The continuous inf-sup condition give us $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that

$$\begin{aligned} \|g^{n+1} - g^n\|_0^2 &= \left\langle g^{n+1} - g^n \,,\, \text{div}\, \mathbf{w} \right\rangle \\ &\le C \left(\frac{\|\mathbf{G}^{n+1} - 2\mathbf{G}^n + \mathbf{G}^{n-1}\|_0}{\triangle t} + \|\mathbf{R}_{n+1} - \mathbf{R}_n\|_{-1} \right. \\ &\qquad \left. + \frac{1}{Re}\left\|\nabla(\mathbf{G}^{n+1} - \mathbf{G}^n)\right\|_0 \right) \|\mathbf{w}\|_1 \end{aligned} \tag{5.4.15}$$

and

$$\begin{aligned} \|g^{n+1} - g^n\|_0 &\le \frac{C}{\beta}\frac{\|\mathbf{G}^{n+1} - 2\mathbf{G}^n + \mathbf{G}^{n-1}\|_0}{\triangle t} + \frac{C}{\beta}\|\mathbf{R}_{n+1} - \mathbf{R}_n\|_{-1} \\ &\quad + \frac{C}{\beta Re}\left\|\nabla(\mathbf{G}^{n+1} - \mathbf{G}^n)\right\|_0. \end{aligned} \tag{5.4.16}$$

Since

$$\|\mathbf{R}_{n+1} - \mathbf{R}_n\|_{-1}^2 \le \triangle t \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt, \tag{5.4.17}$$

squaring the estimate for $\|g^{n+1} - g^n\|_0$ and multiplying $\triangle t$ yields

$$\begin{aligned} \triangle t\left\|g^{n+1} - g^n\right\|_0^2 &\le \frac{C}{\beta^2}\frac{\|\mathbf{G}^{n+1} - 2\mathbf{G}^n + \mathbf{G}^{n-1}\|_0^2}{\triangle t} \\ &+ \frac{C\triangle t}{\beta^2 Re^2}\left\|\nabla(\mathbf{G}^{n+1} - \mathbf{G}^n)\right\|_0^2 + \frac{C}{\beta^2}\triangle t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt. \end{aligned} \tag{5.4.18}$$

On adding over n from 1 to N, the formulas (5.4.3) and (5.4.18) lead to (5.4.5).

■

Lemma 5.12 *Let Assumptions 1-6 hold, and let the exact solution of (1.1.1) satisfy*

$$lemma : error : discret : exact : bound : assump (\mathbf{u}(t_{n+1}), p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega). \tag{5.4.19}$$

If we have

$$\|\nabla \mathbf{u}_t(0)\|_0 \le M, \tag{5.4.20}$$

then

$$\sum_{n=0}^{N} \left\|\mathbf{F}^{n+1} - \mathbf{F}^n\right\|_0^2 + \triangle t \sum_{n=0}^{N} \left\|\nabla\left(\mathbf{F}^{n+1} - \mathbf{F}^n\right)\right\|_0^2 \le C(\triangle t^3 + \triangle t^2 h^2 + \triangle t h^4) \tag{5.4.21}$$

and

$$\sup_{0\le n\le N} \|\eta^n\|^2 \le C(\triangle t + h^{2s}). \tag{5.4.22}$$

Proof. We note Lemma 1.6 which is

$$\int_{t_0}^{t_{N+1}} \left(\|\mathbf{u}_{tt}(t)\|_0^2 + \|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) dt \le C \tag{5.4.23}$$

under assumption (5.4.20). Also we note

$$\left\|\mathbf{F}^{n+1} - \mathbf{F}^n\right\|_0^2 \le 2\left\|\mathbf{G}^{n+1} - \mathbf{G}^n\right\|_0^2 + 2\left\|\mathbf{G}_h^{n+1} - \mathbf{G}_h^n\right\|_0^2 \tag{5.4.24}$$

and we prove in Lemma 5.11,

$$\left\|\mathbf{G}^{n+1} - \mathbf{G}^n\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=1}^{N} \left\|\nabla(\mathbf{G}^{n+1} - \mathbf{G}^n)\right\|_0^2 \le C\triangle t^3. \tag{5.4.25}$$

By (5.3.15),

$$\begin{aligned}\left\|\mathbf{G}_h^{n+1} - \mathbf{G}_h^n\right\|_0^2 &\le Ch^4\left(\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_2^2 + \|p(t_{n+1}) - p(t_n)\|_1^2\right) \\ &\le C\triangle t h^4 \int_{t_n}^{t_{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) dt.\end{aligned} \tag{5.4.26}$$

On adding over n from 0 to N,

$$\sum_{n=0}^{N} \left\|\mathbf{G}_h^{n+1} - \mathbf{G}_h^n\right\|_0^2 \le C\triangle t h^4 \int_{t_0}^{t_{N+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) dt. \tag{5.4.27}$$

And

$$
\begin{aligned}
\left\|\nabla(\mathbf{G}_h^{n+1}-\mathbf{G}_h^n)\right\|_0^2 &\leq C\left(\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_2^2+\|p(t_{n+1})-p(t_n)\|_1^2\right) \\
&\leq C\triangle t h^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2+\|p_t(t)\|_1^2\right)dt.
\end{aligned} \tag{5.4.28}
$$

So

$$
\triangle t\sum_{n=0}^{N}\left\|\nabla(\mathbf{G}_h^{n+1}-\mathbf{G}_h^n)\right\|_0^2 \leq C\triangle t^2h^2\int_{t_0}^{t_{N+1}}\left(\|\mathbf{u}_t(t)\|_2^2+\|p_t(t)\|_1^2\right)dt. \tag{5.4.29}
$$

We prove (5.4.21). (5.4.4) and (5.3.17) derive

$$
\|\eta^n\|^2 \;\leq\; 2(\|g^n\|^2+\|g_h^n\|^2)\leq C(\triangle t+h^{2s}). \tag{5.4.30}
$$

Therefore we finish to prove Lemma 5.12. ■

Lemma 5.13 *Let Assumptions 1-6 hold, and let $C_1h^2 \leq \triangle t \leq C_2h^{\frac{d}{2}}$ be valid with arbitrary constant C_1, $C_2 > 0$. Then we have*

$$
\triangle t\sum_{n=0}^{N}\left\|\widehat{\mathbf{E}}^{n+1}\right\|^2 \leq M. \tag{5.4.31}
$$

PROOF. From Theorems 5.7 and 5.10, we have

$$
\triangle t\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \leq C\left(\triangle t+h^{2s}\right) \tag{5.4.32}
$$

and

$$
\triangle t\sum_{n=0}^{N}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \leq C\left(\triangle t^2+h^{2s+2}\right). \tag{5.4.33}
$$

We note $\left|\left|\left|\widehat{\mathbf{E}}^{n+1}\right|\right|\right| = \left\|\widehat{\mathbf{E}}^{n+1}\right\|_{\mathbf{L}^\infty(\Omega)} + \left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_{\mathbf{L}^3(\Omega)}$. Using inverse inequality Lemma 1.12 and formula (5.4.21), we get

$$
\begin{aligned}
\triangle t\sum_{n=0}^{N}\left|\left|\left|\widehat{\mathbf{E}}^{n+1}\right|\right|\right|^2 &\le C\triangle t\sum_{n=0}^{N}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|^2_{\mathbf{L}^\infty(\Omega)} + \left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|^2_{\mathbf{L}^3(\Omega)}\right) \\
&\le C\triangle t\sum_{n=0}^{N}\left(h^{-d}\left\|\widehat{\mathbf{E}}^{n+1}\right\|^2_0 + h^{-\frac{d}{3}}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|^2_0\right) \\
&\le Ch^{-d}(\triangle t^2 + h^{2s+2}) + Ch^{-\frac{d}{3}}(\triangle t + h^2 m) \\
&\le Ch^{-d}\triangle t^2 + Ch^{2s+2-d} + Ch^{2s-\frac{d}{3}} + Ch^{-\frac{d}{3}}\triangle t \\
&\le M. \qquad \blacksquare
\end{aligned}
\tag{5.4.34}
$$

We note that Lemma 5.13 does <u>not</u> imply $\max\limits_{0\le n\le N}\left|\left|\left|\widehat{\mathbf{E}}^{n}\right|\right|\right|^2 \le M$. Since this inequality does not hold, we need to intermediate step to treat convection term in the proof of Lemma 5.15 below.

Lemma 5.14 *Let Assumptions 1-7 hold, and let $h^2 \le C\triangle t$ with arbitrary $C > 0$. If*

$$\left\|\nabla\mathbf{u}_t(0)\right\|_0 \le M, \tag{5.4.35}$$

then we have

$$
\begin{aligned}
&\left\|\mathbf{E}^1\right\|_0^2 + \frac{\triangle t}{Re}\left\|s_h^1\right\|_0^2 + \frac{1}{2}\left\|\mathbf{E}^1 - \mathbf{E}^0\right\|_0^2 + \left\|\nabla\rho_h^1\right\|_0^2 + \frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^1\right\|_0^2 \\
&\qquad \le C\left(\triangle t^2 + h^{2s+2}\right),
\end{aligned}
\tag{5.4.36}
$$

and

$$\left\|\nabla\mathbf{v}_h^1\right\|_0^2 \le C\triangle t\left(\triangle t^2 + h^{2s+2}\right). \tag{5.4.37}$$

PROOF. By choosing $n=0$ in (5.3.43), we have

$$\begin{aligned}
&\left\|\mathbf{E}^{1}\right\|_{0}^{2}+\frac{1}{2}\left\|\mathbf{E}^{1}-\mathbf{E}^{0}\right\|_{0}^{2}+\left\|\nabla \rho_{h}^{1}\right\|_{0}^{2}+\frac{\triangle t}{2Re}\left\|\nabla \widehat{\mathbf{E}}^{1}\right\|_{0}^{2}+\frac{\triangle t}{Re}\left\|s_{h}^{1}\right\|_{0}^{2} \\
\leq &\left\|\mathbf{E}^{0}\right\|_{0}^{2}+\frac{\triangle t}{Re}\left\|s_{h}^{0}\right\|_{0}^{2}+C\left\|\mathbf{F}^{1}\right\|_{0}^{2}+\frac{C\triangle t}{Re}\left\|\nabla \mathbf{F}^{1}\right\|_{0}^{2} \\
&+CRe\triangle t\left(\left\|\mathbf{E}^{0}\right\|_{0}^{2}+\left\|\mathbf{G}_{h}^{1}\right\|_{0}^{2}+\left\|\mathbf{G}^{0}\right\|_{0}^{2}+\left\|\eta^{1}\right\|_{0}^{2}\right) \\
&+C\triangle t^{2}\left(\left\|\mathbf{u}(t_{1})\right\|_{2}^{2}+\left\|\nabla p(t_{1})\right\|_{0}^{2}\right)+CRe\triangle t^{2}\int_{t_{0}}^{t_{1}}\left\|\mathbf{u}_{t}(t)\right\|_{0}^{2}dt.
\end{aligned} \tag{5.4.38}$$

Since $s_h^0=0$, and by Lemmas 5.2-5.4 and 5.12, we get (5.4.36). And by choosing $n=0$ in (5.3.69), we have

$$\begin{aligned}
\left\|\nabla \mathbf{v}_{h}^{1}\right\|_{0}^{2} \leq\; & CRe\triangle th^{2}\left\|\eta^{1}\right\|_{0}^{2}+C\triangle t^{2}\int_{t_{0}}^{t_{1}}\left\|\mathbf{u}_{t}(t)\right\|_{0}^{2}dt \\
&+\frac{C\triangle t}{Re}\left(h^{2}\left\|\nabla \mathbf{F}^{1}\right\|_{0}^{2}+\left\|\mathbf{F}^{1}\right\|_{0}^{2}+\left\|\mathbf{E}^{1}-\mathbf{E}^{0}\right\|_{0}^{2}\right) \\
&+CRe\triangle t\left\|\nabla \mathbf{v}_{h}^{1}\right\|_{0}^{2}+\frac{C\triangle t}{Re}\left(\left\|\mathbf{G}^{1}\right\|_{0}^{2}+\left\|\mathbf{G}^{0}\right\|_{0}^{2}+\left\|\nabla \rho_{h}^{1}\right\|_{0}^{2}\right) \\
&+CRe\triangle t(\triangle t+h^{2s})\left(\left\|\nabla \mathbf{G}^{1}\right\|_{0}^{2}+\left\|\nabla \widehat{\mathbf{E}}^{1}\right\|_{0}^{2}\right).
\end{aligned} \tag{5.4.39}$$

By Lemmas 5.2-5.4 and 5.12, (5.4.36), and $\|\nabla \mathbf{v}_h^1\|_0^2 \leq \|\mathbf{E}^1\|_0^2$, we get (5.4.37). ■

Lemma 5.15 *Let Assumptions 1-7 hold, and* $(\mathbf{u}(t_{n+1}),\ p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$. *If* $C_1h^2 \leq \triangle t \leq C_2 h^{\frac{2d}{3}}$ *with arbitrary* $C_1,\ C_2>0$ *and if*

$$\left\|\nabla \mathbf{u}_t(0)\right\|_0 \leq M, \tag{5.4.40}$$

then we have

$$\begin{aligned}
&\left\|\mathbf{E}^{N+1}-\mathbf{E}^{N}\right\|_{0}^{2}+\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_{0}^{2} \\
&+\sum_{n=1}^{N}\left\|\nabla\left(\rho_{h}^{n+1}-\rho_{h}^{n}\right)\right\|_{0}^{2}+\frac{\triangle t}{Re}\left\|s_{h}^{N+1}-s_{h}^{N}\right\|_{0}^{2} \\
&+\frac{\triangle t}{8Re}\sum_{n=1}^{N}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right)\right\|_{0}^{2} \leq C\left(\triangle t^{2}+h^{2s+2}\right).
\end{aligned} \tag{5.4.41}$$

Proof. By subtracting n-step formula from $(n+1)$-step formula of (5.3.29), we have

$$\left\langle \frac{\widehat{\mathbf{E}}^{n+1}-\mathbf{E}^n}{\triangle t} - \frac{\widehat{\mathbf{E}}^{n}-\mathbf{E}^{n-1}}{\triangle t}\, ,\, \mathbf{w}_h \right\rangle + \frac{1}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\, ,\, \nabla \mathbf{w}_h \right\rangle$$
$$= \left\langle P^{n+1}-P^n\, ,\, \mathrm{div}\ \mathbf{w}_h \right\rangle - \frac{1}{Re}\left\langle s_h^n - s_h^{n-1}\, ,\, \mathrm{div}\ \mathbf{w}_h \right\rangle \tag{5.4.42}$$
$$-\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w}_h) + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h)$$
$$+\mathcal{N}(\mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{w}_h) - \mathcal{N}(\mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^{n}, \mathbf{w}_h).$$

We choose $\mathbf{w}_h = 2\triangle t(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)$ in (5.4.42). The left hand side of (5.4.42) becomes, by Lemma 5.5,

$$2\left\langle (\widehat{\mathbf{E}}^{n+1}-\mathbf{E}^n) - (\widehat{\mathbf{E}}^{n}-\mathbf{E}^{n-1})\, ,\, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n \right\rangle$$
$$= \left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 - \left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0^2 + \left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2 \tag{5.4.43}$$
$$+2\left\|\nabla(\rho_h^{n+1}-\rho_h^n)\right\|_0^2 - 2\left\langle \mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\, ,\, \mathbf{F}^{n+1}-\mathbf{F}^n \right\rangle$$

and

$$\frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\, ,\, \nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n) \right\rangle$$
$$= \frac{2\triangle t}{Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2 \tag{5.4.44}$$
$$-\frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\, ,\, \nabla(\mathbf{F}^{n+1}-\mathbf{F}^n) \right\rangle.$$

By (5.4.43) and (5.4.44), (5.4.42) becomes

$$
\begin{aligned}
&\big\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\big\|_0^2-\big\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\big\|_0^2+\big\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\big\|_0^2\\
&+2\big\|\nabla(\rho_h^{n+1}-\rho_h^{n})\big\|_0^2+\frac{2\triangle t}{Re}\Big\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\Big\|_0^2\\
&=\frac{2\triangle t}{Re}\Big\langle\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\,,\,\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\Big\rangle\\
&\quad+2\left\langle\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\,,\,\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\rangle\\
&\quad+2\triangle t\Big\langle P^{n+1}-P^{n}\,,\,\mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big\rangle\\
&\quad-\frac{2\triangle t}{Re}\Big\langle s_h^{n}-s_h^{n-1}\,,\,\mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big\rangle\\
&\quad-2\triangle t\Big(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\\
&\qquad\qquad-\mathcal{N}(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big)\\
&\quad+2\triangle t\Big(\mathcal{N}(\mathbf{u}(t_{n}),\mathbf{u}(t_{n}),\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})-\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big)\\
&=A_1+A_2+A_3+A_4+A_5+A_6.
\end{aligned}
\tag{5.4.45}
$$

We estimate the split terms in (5.4.45) separately. A_1-A_3 can be derived easily as follows:

$$
A_1\le\frac{\triangle t}{16Re}\Big\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\Big\|_0^2+\frac{C\triangle t}{Re}\big\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\big\|_0^2,\tag{5.4.46}
$$

$$
A_2\le\frac{1}{2}\big\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\big\|_0^2+C\big\|\mathbf{F}^{n+1}-\mathbf{F}^{n}\big\|_0^2,\tag{5.4.47}
$$

and

$$
\begin{aligned}
A_3&=2\triangle t\Big\langle(p(t_{n+1})-p(t_n))-(g^{n+1}-g^{n})\,,\,\mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big\rangle\\
&\le CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2dt+CRe\triangle t\big\|g^{n+1}-g^{n}\big\|_0^2\\
&\quad+\frac{\triangle t}{16Re}\Big\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\Big\|_0^2.
\end{aligned}
\tag{5.4.48}
$$

By Lemma 5.6 and formula (5.1.12) in Algorithm 5.2, we deduce

$$
\begin{aligned}
A_4 &= \frac{2\triangle t}{Re}\left\langle s_h^n - s_h^{n-1}\,,\, \mathrm{div}\,(\widehat{\mathbf{u}}_h^{n+1} - \widehat{\mathbf{u}}_h^n)\right\rangle \\
&= -\frac{2\triangle t}{Re}\left\langle s_h^n - s_h^{n-1}\,,\, (s_h^{n+1} - s_h^n) - (s_h^n - s_h^{n-1})\right\rangle \\
&= -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1} - s_h^n\right\|_0^2 - \left\|s_h^n - s_h^{n-1}\right\|_0^2\right) \\
&\quad +\frac{\triangle t}{Re}\left\|s_h^{n+1} - 2s_h^n + s_h^{n-1}\right\|_0^2 \\
&\leq -\frac{\triangle t}{Re}\left(\left\|s_h^{n+1} - s_h^n\right\|_0^2 - \left\|s_h^n - s_h^{n-1}\right\|_0^2\right) \\
&\quad +\frac{\triangle t}{Re}\left\|\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\right\|_0^2.
\end{aligned}
\tag{5.4.49}
$$

The convection can be separated by several terms,

$$
\begin{aligned}
A_5 &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}_h^n, \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}_h^n, \mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&= A_{5,1} + A_{5,2} + A_{5,3}
\end{aligned}
\tag{5.4.50}
$$

and

$$
\begin{aligned}
A_6 &= 2\triangle t\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1}), \mathbf{u}(t_n), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{u}(t_n), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1}, \mathbf{u}(t_n) - \widehat{\mathbf{u}}_h^n, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \\
&= A_{6,1} + A_{6,2} + A_{6,3}.
\end{aligned}
\tag{5.4.51}
$$

We now estimate the first two terms in (5.4.50) and (5.4.51) as follows,

$$
\begin{aligned}
A_{5,1} &\leq C\triangle t\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla\left(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right)\right\|_0 \\
&\leq CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + \frac{\triangle t}{64Re}\left\|\nabla\left(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right)\right\|_0^2,
\end{aligned}
\tag{5.4.52}
$$

$$
\begin{aligned}
A_{5,2} &\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\le CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2\right)+\frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2,
\end{aligned} \tag{5.4.53}
$$

$$
\begin{aligned}
A_{6,1} &\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n-1})\|_0\|\mathbf{u}(t_n)\|_2\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\le CRe\triangle t^2\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_0^2dt+\frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2,
\end{aligned} \tag{5.4.54}
$$

$$
\begin{aligned}
A_{6.2} &\le C\triangle t\left\|\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1}\right\|_0\|\mathbf{u}(t_n)\|_2\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\le CRe\triangle t\left(\left\|\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{G}^{n-1}\right\|_0^2\right)+\frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2.
\end{aligned} \tag{5.4.55}
$$

Each last term in (5.4.50) and (5.4.51) can be written by

$$
\begin{aligned}
A_{5,3}+A_{6,3} &= -2\triangle t\mathcal{N}(\mathbf{u}_h^n,\widehat{\mathbf{E}}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n) \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}_h^n,\mathbf{G}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{E}}_h^n,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n) \\
&\quad +2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1},\mathbf{G}_h^n,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n) \\
&= B_1+B_2+B_3+B_4.
\end{aligned} \tag{5.4.56}
$$

The Lemma 5.4 help us to derive the following two terms

$$
\begin{aligned}
B_2 &= 2\triangle t\mathcal{N}\left((\mathbf{u}(t_n)-\mathbf{u}_h^n)-\mathbf{u}(t_n),\mathbf{G}_h^{n+1},\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n\right) \\
&\le C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_0|\!|\!|\mathbf{G}_h^{n+1}|\!|\!|\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\quad +C\triangle t\|\mathbf{u}(t_n)\|_2\left\|\mathbf{G}_h^{n+1}\right\|_0\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\le CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2\right) \\
&\quad +\frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2,
\end{aligned} \tag{5.4.57}
$$

and

$$\begin{aligned}
B_4 &= 2\triangle t\mathcal{N}\left((\mathbf{u}_h^{n-1} - \mathbf{u}(t_{n-1})) + \mathbf{u}(t_{n-1}), \mathbf{G}_h^n, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right) \\
&\le C\triangle t\left\|\mathbf{u}_h^{n-1} - \mathbf{u}(t_{n-1})\right\|_0 \|\mathbf{G}_h^n\| \left\|\nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\quad + C\triangle t\|\mathbf{u}(t_{n-1})\|_2 \|\mathbf{G}_h^n\|_0 \left\|\nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\|_0 \\
&\le CRe\triangle t\left(\left\|\mathbf{E}^{n-1}\right\|_0^2 + \left\|\mathbf{G}^{n-1}\right\|_0^2 + \|\mathbf{G}_h^n\|_0^2\right) \\
&\quad + \frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\|_0^2.
\end{aligned} \tag{5.4.58}$$

Invoking the crucial properties of $\mathcal{N}$ of (1.2.61), we infer that

$$\begin{aligned}
B_1 + B_3 &= 2\triangle t\left(\mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n) + \mathcal{N}(\mathbf{u}_h^{n-1}, \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^{n+1})\right) \\
&= 2\triangle t\mathcal{N}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n) \\
&= -2\triangle t\mathcal{N}(\mathbf{E}^n - \mathbf{E}^{n-1}, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^n) \\
&\quad - 2\triangle t\mathcal{N}(\mathbf{G}^n - \mathbf{G}^{n-1}, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^n) \\
&\quad + 2\triangle t\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1}), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^n) \\
&= B_5^n + B_6 + B_7.
\end{aligned} \tag{5.4.59}$$

The term B_5^n can be written by

$$B_5^n = -2\triangle t\mathcal{N}(\mathbf{E}^n - \mathbf{E}^{n-1}, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^n), \tag{5.4.60}$$

and we postpone the estimate of B_5^n until the end of proof. In order to estimate B_5 in end of this proof, The inequality $\left\|\widehat{\mathbf{E}}_h^n\right\|_1 \le C$ and Lemma 5.7 derive the following two terms

$$\begin{aligned}
B_6 &\le C\triangle t\left\|\mathbf{G}^n - \mathbf{G}^{n-1}\right\|_1 \left\|\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right\|_1 \left\|\widehat{\mathbf{E}}_h^n\right\|_1 \\
&\le CRe\triangle t\left\|\nabla(\mathbf{G}^n - \mathbf{G}^{n-1})\right\|_0^2 + \frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\|_1^2
\end{aligned} \tag{5.4.61}$$

and

$$
\begin{aligned}
B_7 &\leq C\triangle t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n-1})\|_1\left\|\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n\right\|_1\left\|\widehat{\mathbf{E}}_h^n\right\|_1 \\
&\leq CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_1^2dt+\frac{\triangle t}{64Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2.
\end{aligned} \tag{5.4.62}
$$

Gathering (5.4.50)-(5.4.62), A_5+A_6 can be bounded by

$$
\begin{aligned}
A_5+A_6 \leq\ & CRe\triangle t\Big(\|\mathbf{E}^n\|_0^2+\left\|\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2+\|\mathbf{G}^n\|_0^2 \\
&+\left\|\mathbf{G}^{n-1}\right\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2+\|\mathbf{G}_h^n\|_0^2+\left\|\nabla(\mathbf{G}^n-\mathbf{G}^{n-1})\right\|_0^2\Big) \\
&+\frac{\triangle t}{4Re}\left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\right\|_0^2\right) \\
&+CRe\triangle t^2\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_t(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2\right)dt+B_5^n,
\end{aligned} \tag{5.4.63}
$$

since

$$
\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^n)\right\|_0^2 \leq 2\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+2\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\right\|_0^2. \tag{5.4.64}
$$

Upon plugging (5.4.46)-(5.4.49) and (5.4.63) into (5.4.45), we get

$$
\begin{aligned}
&\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2-\left\|\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2+\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}-s_h^n\right\|_0^2-\left\|s_h^n-s_h^{n-1}\right\|_0^2\right) \\
&+\frac{1}{2}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2+\left\|\nabla\left(\rho_h^{n+1}-\rho_h^n\right)\right\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right)\right\|_0^2 \\
&\leq CRe\triangle t\Big(\|\mathbf{E}^n\|_0^2+\left\|\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2+\|\mathbf{G}_h^n\|_0^2 \\
&\quad+\|\mathbf{G}^n\|_0^2+\left\|\mathbf{G}^{n-1}\right\|_0^2+\left\|\nabla(\mathbf{G}^n-\mathbf{G}^{n-1})\right\|_0^2+\left\|g^{n+1}-g^n\right\|_0^2\Big) \\
&\quad+CRe\triangle t^2\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_t(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|p_t(t)\|_0^2\right)dt \\
&\quad+\frac{C\triangle t}{Re}\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\right\|_0^2+C\left\|\mathbf{F}^{n+1}-\mathbf{F}^n\right\|_0^2+B_5^n.
\end{aligned}
$$

Summing over n from 0 to N, and noting $s_h^0 = 0$, implies

$$
\begin{aligned}
&\left\|\mathbf{E}^{N+1}-\mathbf{E}^{N}\right\|_0^2+\frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2+\frac{\triangle t}{Re}\left\|s_h^{N+1}-s_h^{N}\right\|_0^2\\
&+\sum_{n=1}^{N}\left\|\nabla(\rho_h^{n+1}-\rho_h^{n})\right\|_0^2+\frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2\\
&\le\sum_{n=1}^{N}B_5^n+\left\|\mathbf{E}^{1}-\mathbf{E}^{0}\right\|_0^2+\frac{\triangle t}{Re}\left\|s_h^1\right\|_0^2+C\sum_{n=1}^{N}\left\|\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\|_0^2\\
&\quad+CRe\triangle t\sum_{n=0}^{N+1}\Big(\|\mathbf{E}^n\|_0^2+\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0^2+\left\|g^{n+1}-g^{n}\right\|_0^2+\|\mathbf{G}_h^n\|_0^2\\
&\quad+\|\mathbf{G}^n\|_0^2+\left\|\nabla(\mathbf{G}^{n}-\mathbf{G}^{n-1})\right\|_0^2\Big)+\frac{C\triangle t}{Re}\sum_{n=1}^{N}\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\right\|_0^2\\
&\quad+CRe\triangle t^2\int_{t_0}^{t_{N+1}}\left(\|\mathbf{u}_t(t)\|_0^2+\|\mathbf{u}_t(t)\|_1^2+\|p_t(t)\|_0^2\right)dt.
\end{aligned}
\tag{5.4.65}
$$

By Lemmas 5.2-5.4, 5.7 5.10-5.12, and 5.14, we deduce

$$
\begin{aligned}
&\left\|\mathbf{E}^{N+1}-\mathbf{E}^{N}\right\|_0^2+\frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2+\frac{\triangle t}{Re}\left\|s_h^{N+1}-s_h^{N}\right\|_0^2\\
&\quad+\sum_{n=1}^{N}\left\|\nabla\left(\rho_h^{n+1}-\rho_h^{n}\right)\right\|_0^2+\frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2\\
&\qquad\le C(\triangle t^2+h^{2s+2})+\sum_{n=1}^{N}B_5^n.
\end{aligned}
\tag{5.4.66}
$$

We now estimate the remainder B_5^n. Since $\left\|\widehat{\mathbf{E}}_h^n\right\|_1 \le C$, by Lemmas 1.12 and 1.16, we have

$$
\begin{aligned}
B_5^n &\le C\triangle t\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_{\mathbf{L}^3(\Omega)}\left\|\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n}\right\|_1\left\|\widehat{\mathbf{E}}_h^{n}\right\|_1\\
&\le C\triangle t h^{-\frac{d}{6}}\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\right\|_0\\
&\le CRe\triangle t h^{-\frac{d}{3}}\left\|\mathbf{E}^{n}-\mathbf{E}^{n-1}\right\|_0^2+\frac{\triangle t}{8Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\right\|_0.
\end{aligned}
\tag{5.4.67}
$$

The result, $\sum_{n=1}^{N} \left\| \mathbf{E}^n - \mathbf{E}^{n-1} \right\|_0^2 \leq C(\triangle t + h^{2s})$, in Lemma 5.7 implies

$$\sum_{n=1}^{N} B_5^n \leq \frac{\triangle t}{4Re} \sum_{n=1}^{N} \left\| \nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n) \right\|_0^2 + CRe(\triangle t^2 h^{-\frac{d}{3}} + \triangle t h^{2s-\frac{d}{3}}). \tag{5.4.68}$$

So (5.4.66) can be bounded by

$$\begin{aligned} &\left\| \mathbf{E}^{N+1} - \mathbf{E}^N \right\|_0^2 + \frac{1}{2} \sum_{n=1}^{N} \left\| \mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} \right\|_0^2 + \frac{\triangle t}{Re} \left\| s_h^{N+1} - s_h^N \right\|_0^2 \\ &\quad + \sum_{n=1}^{N} \left\| \nabla \left(\rho_h^{n+1} - \rho_h^n \right) \right\|_0^2 + \frac{\triangle t}{4Re} \sum_{n=1}^{N} \left\| \nabla \left(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n \right) \right\|_0^2 \\ &\leq C(\triangle t^2 h^{-\frac{d}{3}} + \triangle t h^{2s-\frac{d}{3}}). \end{aligned} \tag{5.4.69}$$

Since it is not yet correct order, we estimate B_5^n again, but this time employing (5.4.69). We now use the improved estimates

$$\left\| \mathbf{E}^{N+1} - \mathbf{E}^N \right\|_0^2 \leq C(\triangle t^2 h^{-\frac{d}{3}} + \triangle t h^{2s-\frac{d}{3}}). \tag{5.4.70}$$

We see that

$$\begin{aligned} B_5^n &\leq C\triangle t \left\| \mathbf{E}^n - \mathbf{E}^{n-1} \right\|_{\mathbf{L}^3(\Omega)} \left\| \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n \right\|_1 \left\| \widehat{\mathbf{E}}_h^n \right\|_1 \\ &\leq C\triangle t h^{-\frac{d}{6}} \left\| \mathbf{E}^n - \mathbf{E}^{n-1} \right\|_0 \left\| \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n \right\|_1 \left\| \widehat{\mathbf{E}}_h^n \right\|_1 \\ &\leq C\triangle t h^{-\frac{d}{3}} (\triangle t + \triangle t^{\frac{1}{2}} h^s) \left\| \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n \right\|_1 \left\| \widehat{\mathbf{E}}_h^n \right\|_1 \\ &\leq CRe\triangle t^2 (\triangle t h^{-\frac{2d}{3}} + h^{2s-\frac{2d}{3}}) \left\| \nabla \widehat{\mathbf{E}}_h^n \right\|_0^2 \\ &\quad + \frac{\triangle t}{4Re} \left\| \nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \right\|_0^2. \end{aligned} \tag{5.4.71}$$

Since $\triangle t \leq C_2 h^{\frac{2d}{3}}$, we obtain

$$\sum_{n=1}^{N} B_5^n \leq CRe\triangle t^2 \sum_{n=1}^{N} \left\| \nabla \widehat{\mathbf{E}}_h^n \right\|_0^2 + \frac{\triangle t}{4Re} \sum_{n=1}^{N} \left\| \nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) \right\|_0^2. \tag{5.4.72}$$

Lemma 5.10, in conjunction with (5.4.66) and (5.4.72) implies (5.4.41). ■

Lemma 5.16 *Let Assumptions 1-7 hold. Let* $(\mathbf{u}(t_{n+1}),\ p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$*. If there exist two positive constants* C_1 *and* C_2 *such that* $C_1 h^2 \leq \triangle t \leq C_2 h^{\frac{2d}{3}}$ *for arbitrary* $C_1,\ C_2 > 0$*, and if*

$$\|\nabla \mathbf{u}_t(0)\|_0 \leq M, \tag{5.4.73}$$

then we have

$$\begin{aligned}&\left\|\mathbf{E}^{N+1} - \mathbf{E}^N\right\|^2_{\mathbf{Z}^*} + \frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right\|^2_{\mathbf{Z}^*}\\ &+\frac{\triangle t}{Re}\sum_{n=1}^{N}\left\|\mathbf{E}_h^{n+1} - \mathbf{E}_h^n\right\|_0^2 \leq C\triangle t\left(\triangle t^2 + h^{2s+2}\right).\end{aligned} \tag{5.4.74}$$

Proof. By choose $\mathbf{w}_h = 2\triangle t\left(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right)$ in (5.4.42), where $\mathbf{v}_h^{n+1}$ is the solution of the discrete weak Stokes system (5.3.46). Then we have

$$\begin{aligned}&2\left\langle(\mathbf{E}^{n+1} - \mathbf{E}^n) - (\mathbf{E}^n - \mathbf{E}^{n-1})\,,\,\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right\rangle\\ &+\frac{2\triangle t}{Re}\left\langle\nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\,,\,\nabla\left(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right)\right\rangle\\ &= 2\triangle t\left\langle P^{n+1} - P^n\,,\,\mathrm{div}\,(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\rangle\\ &\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right.\\ &\qquad\left. - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right)\\ &\quad +2\triangle t\left(\mathcal{N}(\mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{v}_h^{n+1} - \mathbf{v}_h^n) - \mathcal{N}(\mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^n, \mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right).\end{aligned} \tag{5.4.75}$$

Here, we note again $\left\langle r_h\,,\,\mathrm{div}\,\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right\rangle = 0$ for all $r_h \in \mathbb{P}_h$. The left-hand side of (5.4.75) become

$$\begin{aligned}&2\left\langle(\mathbf{E}^{n+1} - \mathbf{E}^n) - (\mathbf{E}^n - \mathbf{E}^{n-1})\,,\,\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right\rangle\\ &= \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 - \left\|\nabla(\mathbf{v}_h^n - \mathbf{v}_h^{n-1})\right\|_0^2\\ &\quad+\left\|\nabla(\mathbf{v}_h^{n+1} - 2\mathbf{v}_h^n + \mathbf{v}_h^{n-1})\right\|_0^2\end{aligned} \tag{5.4.76}$$

because of (5.3.46), and

$$
\begin{aligned}
&\frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\,,\,\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\rangle \\
&=\frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})+\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\,,\,\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\rangle \\
&=\frac{2\triangle t}{Re}\left(\left\langle \mathbf{E}^{n+1}-\mathbf{E}^{n}\,,\,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n}\right\rangle-\left\langle \nabla\left(q_h^{n+1}-q_h^{n}\right),\,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n}\right\rangle\right) \\
&\quad+\frac{2\triangle t}{Re}\left\langle \nabla\left(\mathbf{F}^{n+1}-\mathbf{F}^{n}\right),\,\nabla\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\rangle \qquad (5.4.77)\\
&=\frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2-\frac{2\triangle t}{Re}\left\langle \mathbf{E}^{n+1}-\mathbf{E}^{n}\,,\,\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\rangle \\
&\quad-\frac{2\triangle t}{Re}\left\langle \nabla\left(q_h^{n+1}-q_h^{n}\right),\,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n}\right\rangle \\
&\quad+\frac{2\triangle t}{Re}\left\langle \nabla\left(\mathbf{F}^{n+1}-\mathbf{F}^{n}\right),\,\nabla\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\rangle .
\end{aligned}
$$

because of (5.3.25) and the weak formula (5.3.46). So (5.4.75) can be rewritten as follows:

$$
\begin{aligned}
&\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\|_0^2-\left\|\nabla(\mathbf{v}_h^{n}-\mathbf{v}_h^{n-1})\right\|_0^2+\frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2 \\
&+\left\|\nabla(\mathbf{v}_h^{n+1}-2\mathbf{v}_h^{n}+\mathbf{v}_h^{n-1})\right\|_0^2=\frac{2\triangle t}{Re}\left\langle \mathbf{E}^{n+1}-\mathbf{E}^{n}\,,\,\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\rangle \\
&\qquad+\frac{2\triangle t}{Re}\left\langle \nabla(q_h^{n+1}-q_h^{n})\,,\,\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n}\right\rangle \\
&\qquad-\frac{2\triangle t}{Re}\left\langle \nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\,,\,\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\rangle \\
&\qquad+2\triangle t\left\langle P^{n+1}-P^{n}\,,\,\mathrm{div}\,(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\rangle \qquad (5.4.78)\\
&\qquad-2\triangle t\quad\big(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}) \\
&\qquad\qquad-\mathcal{N}(\mathbf{u}(t_n),\mathbf{u}(t_n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}) \\
&\qquad\qquad-\mathcal{N}(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}) \\
&\qquad\qquad+\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\big)\,. \\
&=A_1+A_2+A_3+A_4+A_5.
\end{aligned}
$$

The first term in (5.4.78) is simply deduced by

$$A_1 \leq \frac{\triangle t}{4Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{C\triangle t}{Re}\left\|\mathbf{F}^{n+1}-\mathbf{F}^n\right\|_0^2. \tag{5.4.79}$$

We note Assumption 1 which in this context reads

$$\left\|\mathbf{v}^{n+1}\right\|_2+\left\|q^{n+1}\right\|_1 \leq C\left\|\mathbf{E}^{n+1}\right\|_0. \tag{5.4.80}$$

By Lemma 5.5, we have

$$\begin{aligned} A_2 &= -\frac{2\triangle t}{Re}\left\langle\nabla(q_h^{n+1}-q_h^n)\,,\,\nabla(\rho_h^{n+1}-\rho_h^n)\right\rangle \\ &\leq \frac{\triangle t}{4Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{C\triangle t}{Re}\left\|\nabla(\rho_h^{n+1}-\rho_h^n)\right\|_0^2, \end{aligned} \tag{5.4.81}$$

and

$$\begin{aligned} A_3 &= -\frac{2\triangle t}{Re}\left\langle\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\,,\,\nabla(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\rangle \\ &\quad+\frac{2\triangle t}{Re}\left\langle\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\,,\,\nabla(\mathbf{v}^{n+1}-\mathbf{v}_h^{n+1})-\nabla(\mathbf{v}^n-\mathbf{v}_h^n)\right\rangle \\ &\leq \frac{C\triangle t}{Re}\left(\left\|\mathbf{F}^{n+1}-\mathbf{F}^n\right\|_0+h\left\|\nabla\left(\mathbf{F}^{n+1}-\mathbf{F}^n\right)\right\|_0\right)\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2 \\ &\leq \frac{\triangle t}{4Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{C\triangle t}{Re}\left\|\mathbf{F}^{n+1}-\mathbf{F}^n\right\|_0^2 \\ &\quad+\frac{C\triangle t h^2}{Re}\left\|\nabla\left(\mathbf{F}^{n+1}-\mathbf{F}^n\right)\right\|_0^2. \end{aligned} \tag{5.4.82}$$

Since div $\mathbf{v}^{n+1}=0$, we obtain

$$\begin{aligned} A_4 &= 2\triangle t\left\langle P^{n+1}-P^n\,,\,\mathrm{div}\,(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)-\mathrm{div}\,(\mathbf{v}^{n+1}-\mathbf{v}^n)\right\rangle \\ &\leq C\triangle t h\left\|P^{n+1}-P^n\right\|_0\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2 \\ &\leq C Re\triangle t^2 h^2\int_{t_n}^{t_{n+1}}\|p_t(t)\|_0^2 dt+C Re\triangle t h^2\left\|g^{n+1}-g^n\right\|_0^2 \\ &\quad+\frac{\triangle t}{4Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2. \end{aligned} \tag{5.4.83}$$

The convection A_5 can be split as follows

$$\begin{aligned} A_5 \ &= \ -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n),\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ &\quad +2\triangle t\mathcal{N}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ &\quad +2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n,\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \end{aligned} \tag{5.4.84}$$

whence

$$\begin{aligned} A_5 = \ & -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1})-2\mathbf{u}(t_n)+\mathbf{u}(t_{n-1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ & -2\triangle t\mathcal{N}((\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))-(\mathbf{u}_h^n-\mathbf{u}_h^{n-1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ & -2\triangle t\mathcal{N}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ & -2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ & -2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1},\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ & -2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1},(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-(\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\ = \ & B_1+B_2+B_3+B_4+B_5+B_6. \end{aligned} \tag{5.4.85}$$

Since $\|\mathbf{u}(t_{n+1})\|_2 \le C$, we get

$$\begin{aligned} B_1 \ &\le \ C\triangle t\|\mathbf{u}(t_{n+1})-2\mathbf{u}(t_n)+\mathbf{u}(t_{n-1})\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\mathbf{v}_h^{n+1}-\mathbf{v}_h^n\right\|_1 \\ &\le \ C\triangle t^4\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt + C\triangle t\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2 \end{aligned} \tag{5.4.86}$$

and

$$\begin{aligned} B_2 \ &= \ -2\triangle t\mathcal{N}\left((\mathbf{G}^n-\mathbf{G}^{n-1})+(\mathbf{E}^n-\mathbf{E}^{n-1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n\right) \\ &\le \ \frac{\triangle t}{20Re}\left(\left\|\mathbf{G}^n-\mathbf{G}^{n-1}\right\|_0^2+\left\|\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2\right) \\ &\quad +CRe\triangle t\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2. \end{aligned} \tag{5.4.87}$$

B_3 can be divided by

$$
\begin{aligned}
B_3 &= 2\triangle t\mathcal{N}((\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))-(\mathbf{u}_h^n-\mathbf{u}_h^{n-1}),\\
&\qquad \mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\\
&\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^n\right)\\
&= 2\triangle t\mathcal{N}((\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))-(\mathbf{u}_h^n-\mathbf{u}_h^{n-1}),\\
&\qquad \mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)-(\mathbf{v}^{n+1}-\mathbf{v}^n))\\
&\quad +2\triangle t\mathcal{N}((\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))-(\mathbf{u}_h^n-\mathbf{u}_h^{n-1}),\\
&\qquad \mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}^{n+1}-\mathbf{v}^n)\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\\
&\qquad \mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)-(\mathbf{v}^{n+1}-\mathbf{v}^n))\\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}),\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}^{n+1}-\mathbf{v}^n)\\
&= B_{3,1}+B_{3,2}+B_{3,3}+B_{3,4}.
\end{aligned} \tag{5.4.88}
$$

Since we have $\left\|(\mathbf{G}^n - \mathbf{G}^{n-1}) + (\mathbf{E}^n - \mathbf{E}^{n-1})\right\|_0 \leq C(\triangle t + h^{s+1})$ by Lemmas 5.2 and 5.15, we get

$$
\begin{aligned}
B_{3,1} &\leq C\triangle th\left\|(\mathbf{G}^n - \mathbf{G}^{n-1}) + (\mathbf{E}^n - \mathbf{E}^{n-1})\right\|_{\mathbf{L}^3(\Omega)} \\
&\quad \left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\leq C\triangle th^{1-\frac{d}{6}}\left\|(\mathbf{G}^n - \mathbf{G}^{n-1}) + (\mathbf{E}^n - \mathbf{E}^{n-1})\right\|_0 \\
&\quad \left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \qquad (5.4.89)\\
&\leq C\triangle th^{\frac{1}{2}}(\triangle t + h^{s+1})\left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \\
&\leq CRe(\triangle t^3 h + \triangle t h^{2s+3})\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right) \\
&\quad + \frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2
\end{aligned}
$$

and

$$
\begin{aligned}
B_{3,2} &\leq C\triangle t\left\|(\mathbf{G}^n - \mathbf{G}^{n-1}) + (\mathbf{E}^n - \mathbf{E}^{n-1})\right\|_0 \\
&\quad \left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\leq C\triangle t(\triangle t + h^{s+1})\left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \qquad (5.4.90)\\
&\leq CRe(\triangle t^3 + \triangle t h^{2s+2})\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right) \\
&\quad + \frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2.
\end{aligned}
$$

Since $\left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1 \leq C\left(\left\|\nabla\mathbf{G}^{n+1}\right\|_0 + \left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\right) \leq C$ by Lemmas 5.2 and 5.7, we obtain

$$
\begin{aligned}
B_{3,3} &\leq C\triangle th\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1\left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\
&\leq C\triangle th\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \qquad (5.4.91)\\
&\leq C\triangle t^2 h^2\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_t(t)\|_0^2 dt + \frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2.
\end{aligned}
$$

Lemmas 5.2 and 5.7 implies $\left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_0 \le C(\triangle t^{\frac{1}{2}} + h^s)$ and

$$\begin{aligned} B_{3,4} &\le C\triangle t \|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1 \left\|\mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_0 \left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\ &\le C\triangle t(\triangle t^{\frac{1}{2}} + h^s)\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1 \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0 \\ &\le C(\triangle t^3 + \triangle t^2 h^{2s}) \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_t(t)\|_0^2 dt + \frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2. \end{aligned} \tag{5.4.92}$$

Since $\|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1 \le C\left(\triangle t \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_1^2 dt\right)^{\frac{1}{2}} \le C\triangle t^{\frac{1}{2}}$, we deduce

$$\begin{aligned} B_4 &\le C\triangle t \|\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})\|_1 \|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1 \left\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right\|_1 \\ &\le C\triangle t^3 \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_t(t)\|_0^2 dt + C\triangle t \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2. \end{aligned} \tag{5.4.93}$$

We divide B_5 term as the following

$$\begin{aligned} B_5 &= 2\triangle t \mathcal{N}(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \\ &\qquad (\mathbf{v}^{n+1} - \mathbf{v}^n) - (\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)) \\ &\quad -2\triangle t \mathcal{N}(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{v}^{n+1} - \mathbf{v}^n) \\ &= B_{5,1} + B_{5,2}. \end{aligned} \tag{5.4.94}$$

Since $\left\|\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}\right\|_{\mathbf{L}^3(\Omega)} \le Ch^{-\frac{d}{6}}(\triangle t^{\frac{1}{2}} + h^s)$, we obtain two followings

$$\begin{aligned} B_{5,1} &\le C\triangle t h \left\|\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}\right\|_{\mathbf{L}^3(\Omega)} \|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1 \\ &\qquad \left\|\mathbf{v}^{n+1} - \mathbf{v}^n\right\|_2 \\ &\le C\triangle t^2(\triangle t h + h^{2s+1}) \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_t(t)\|_0^2 dt \\ &\quad + \frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2, \end{aligned} \tag{5.4.95}$$

and

$$
\begin{aligned}
B_{5,2} &\leq C\triangle t\left\|\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1}\right\|_0\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_1\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2 \\
&\leq C\triangle t(\triangle t^{\frac{1}{2}}+h^s)\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_1\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0 \qquad (5.4.96)\\
&\leq C\triangle t^2(\triangle t+h^{2s})\int_{t_n}^{t_{n+1}}\|\nabla\mathbf{u}_t(t)\|_0^2dt+\frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2.
\end{aligned}
$$

We now estimate the last term in (5.4.85)

$$
\begin{aligned}
B_6 = &-2\triangle t\mathcal{N}(\mathbf{u}_h^{n-1},(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-(\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n) \\
= &\ 2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1},(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-(\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n),\\
&\qquad (\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)-(\mathbf{v}^{n+1}-\mathbf{v}^n))\\
&+2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1},(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-(\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n), \qquad (5.4.97)\\
&\qquad \mathbf{v}^{n+1}-\mathbf{v}^n)\\
&-2\triangle t\mathcal{N}(\mathbf{u}(t_{n-1}),(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-(\widehat{\mathbf{u}}_h^{n+1}-\widehat{\mathbf{u}}_h^n),\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\\
= &\ B_{6,1}+B_{6,2}+B_{6,3}.
\end{aligned}
$$

Then each split term is proved by

$$
\begin{aligned}
B_{6,1} &\leq C\triangle th\left\|\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1}\right\|_{\mathbf{L}^3(\Omega)}\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2\\
&\quad\left\|(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)+(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_1\\
&\leq C(\triangle t^{\frac{3}{2}}h^{\frac{1}{2}}+\triangle th^{s+\frac{1}{2}})\left\|(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)+(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_1\\
&\quad\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0 \qquad (5.4.98)\\
&\leq C\triangle t(\triangle th+h^{2s+1})\left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_0^2\right)\\
&\quad+\frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2,
\end{aligned}
$$

$$
\begin{aligned}
B_{6,2} \ \leq\ & C\triangle t\left\|\mathbf{u}(t_{n-1})-\mathbf{u}_h^{n-1}\right\|_0\left\|\mathbf{v}^{n+1}-\mathbf{v}^n\right\|_2 \\
& \left\|(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)+(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_1 \\
\leq\ & C\triangle t(\triangle t^{\frac{1}{2}}+h^s)\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0 \\
& \left\|(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)+(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_1 \\
\leq C\triangle t(\triangle t+h^{2s}) & \left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_0^2\right) \\
+\frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2, &
\end{aligned}
\tag{5.4.99}
$$

and

$$
\begin{aligned}
B_{6,3} \ \leq\ & C\triangle t\|\mathbf{u}(t_{n-1})\|_2\left\|\mathbf{v}_h^{n+1}-\mathbf{v}_h^n\right\|_1 \\
& \left\|(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)+(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_0 \\
\leq\ & C\triangle t\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2 \\
& +\frac{\triangle t}{20Re}\left(\left\|\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n\right\|_0^2+\left\|\mathbf{G}^{n+1}-\mathbf{G}^n\right\|_0^2\right).
\end{aligned}
\tag{5.4.100}
$$

Gathering (5.4.84)-(5.4.100) implies

$$
\begin{aligned}
A_5 \leq\ & C\triangle t\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+\frac{\triangle t}{2Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 \\
& +\frac{\triangle t}{20Re}\left(\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2+\left\|\nabla\rho_h^{n+1}-\nabla\rho_h^n\right\|_0^2\right) \\
& +\frac{\triangle t}{20Re}\left(\left\|\mathbf{G}^{n+1}-\mathbf{G}^n\right\|_0^2+\left\|\mathbf{G}^n-\mathbf{G}^{n-1}\right\|_0^2\right) \\
& +CRe(\triangle t^3+\triangle t h^{2s+2})\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right) \\
& +C(\triangle t^3+\triangle t^2h^2)\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\nabla\mathbf{u}_t(t)\|_0^2\right)dt \\
+C(\triangle t^2+\triangle t h^{2s}) & \left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_0^2\right).
\end{aligned}
\tag{5.4.101}
$$

By (5.4.78)-(5.4.83) and (5.4.101), we derive

$$
\begin{aligned}
&\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2-\left\|\nabla(\mathbf{v}_h^n-\mathbf{v}_h^{n-1})\right\|_0^2+\frac{\triangle t}{2Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\\
&+\left\|\nabla(\mathbf{v}_h^{n+1}-2\mathbf{v}_h^n+\mathbf{v}_h^{n-1})\right\|_0^2\le\frac{\triangle t}{20Re}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|_0^2\\
&+\frac{C\triangle t}{Re}\Big(\left\|\mathbf{F}^{n+1}-\mathbf{F}^n\right\|_0^2+\left\|\nabla(\rho_h^{n+1}-\rho_h^n)\right\|_0^2+\left\|\mathbf{G}^{n+1}-\mathbf{G}^n\right\|_0^2\\
&+\left\|\mathbf{G}^n-\mathbf{G}^{n-1}\right\|_0^2+\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^n)\right\|_0^2\Big)\\
&+C\triangle t\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+CRe\triangle th^2\left\|g^{n+1}-g^n\right\|_0^2\\
&+C(\triangle t^2+\triangle th^{2s})\left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^n)\right\|_0^2+\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^n)\right\|_0^2\right)\\
&+CRe(\triangle t^3+\triangle th^{2s+2})\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right)\\
&+C(\triangle t^3+\triangle t^2h^2)\int_{t_{n-1}}^{t_{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\nabla\mathbf{u}_t(t)\|_0^2+\|p_t(t)\|_0^2\right)dt.
\end{aligned}
\tag{5.4.102}
$$

Adding over n from 1 to N, we obtain

$$
\begin{aligned}
&\left\|\nabla(\mathbf{v}_h^{N+1}-\mathbf{v}_h^{N})\right\|_0^2+\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}_h^{n+1}-2\mathbf{v}_h^{n}+\mathbf{v}_h^{n-1})\right\|_0^2\\
&+\frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2\le C\triangle t\sum_{n=1}^{N}\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\|_0^2\\
&\quad+\left\|\nabla(\mathbf{v}_h^{1}-\mathbf{v}_h^{0})\right\|_0^2+\frac{C\triangle t}{Re}\sum_{n=1}^{N}\Big(\left\|\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\|_0^2+\left\|\nabla(\rho_h^{n+1}-\rho_h^{n})\right\|_0^2\\
&\quad+\left\|\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\|_0^2+\left\|\mathbf{G}^{n}-\mathbf{G}^{n-1}\right\|_0^2+h^2\left\|\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\right\|_0^2\Big)\\
&\quad+CRe\triangle th^2\sum_{n=1}^{N}\left\|g^{n+1}-g^{n}\right\|_0^2+\frac{\triangle t}{20Re}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}-2\mathbf{E}^{n}+\mathbf{E}^{n-1}\right\|_0^2\\
&\quad+CRe(\triangle t^3+\triangle th^{2s+2})\sum_{n=1}^{N}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right)\\
&\quad+C(\triangle t^3+\triangle t^2h^2)\int_{t_0}^{t_{N+1}}\left(\|\mathbf{u}_{tt}(t)\|_0^2+\|\nabla\mathbf{u}_t(t)\|_0^2+\|p_t(t)\|_0^2\right)dt\\
&+C(\triangle t^2+\triangle th^{2s})\sum_{n=1}^{N}\left(\left\|\nabla(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n})\right\|_0^2+\left\|\nabla(\mathbf{G}^{n+1}-\mathbf{G}^{n})\right\|_0^2\right).
\end{aligned}
\tag{5.4.103}
$$

By Lemmas 5.9, 5.10, 5.14, 5.12, and 5.15, and by discrete Gronwall lemma, we prove (5.16). ■

We now finally derive the error of pressure by exploiting all previous results.

Lemma 5.17 *Let Assumptions 1-7 hold, and let* $(\mathbf{u}(t_{n+1}),\, p(t_{n+1}))\in\mathbf{H}^{s+1}(\Omega)\times H^s(\Omega)$. *If* $C_1h^2\le\triangle t\le C_2h^{\frac{2d}{3}}$ *for arbitrary* C_1 , $C_2>0$, *and if*

$$\left\|\nabla\mathbf{u}_t(0)\right\|_0\le M, \tag{5.4.104}$$

then we have

$$\triangle t\sum_{n=0}^{N}\left\|e_h^{n+1}\right\|_0^2\le C\left(\triangle t+h^{2s}\right). \tag{5.4.105}$$

PROOF. By the definition of pressure (5.1.14) in Gauge-Uzawa Algorithm, (5.3.29) can be rewritten as

$$\left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^{n}}{\triangle t}\,,\,\mathbf{w}_h \right\rangle - \left\langle e^{n+1}\,,\,\mathrm{div}\,\mathbf{w}_h \right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{w}_h \right\rangle$$

$$= \frac{1}{Re}\left\langle s_h^{n+1}-s_h^{n}\,,\,\mathrm{div}\,\mathbf{w}_h \right\rangle \tag{5.4.106}$$

$$-\mathcal{N}\left(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{w}_h\right) + \mathcal{N}\left(\mathbf{u}_h^n,\widehat{\mathbf{u}}_h^{n+1},\mathbf{w}_h\right).$$

By inf-sup Assumption 4, there exists an element $\mathbf{z}_h^{n+1}\in\mathbb{V}_h$ such that

$$\left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e_h^{n+1}\right\rangle = \left\|e_h^{n+1}\right\|_0^2 \quad \text{and} \quad \left\|\mathbf{z}_h^{n+1}\right\|_1 \le \frac{1}{\beta}\left\|e_h^{n+1}\right\|_0. \tag{5.4.107}$$

the formulas (5.4.106) and (5.4.107) imply

$$\begin{aligned}
\left\|e_h^{n+1}\right\|_0^2 &= \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e_h^{n+1}\right\rangle \\
&= \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e^{n+1}+\eta^{n+1}\right\rangle \\
&= \left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^{n}}{\triangle t}\,,\,\mathbf{z}_h^{n+1}\right\rangle \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{z}_h^{n+1}\right\rangle \\
&\quad -\frac{1}{Re}\left\langle s_h^{n+1}-s_h^{n}\,,\,\mathrm{div}\,\mathbf{z}_h^{n+1}\right\rangle + \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,\eta^{n+1}\right\rangle \\
&\quad + \left(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{z}_h^{n+1}) - \mathcal{N}(\mathbf{u}_h^n,\widehat{\mathbf{u}}_h^{n+1},\mathbf{z}_h^{n+1})\right) \\
&= A_1+A_2+A_3+A_4+A_5.
\end{aligned} \tag{5.4.108}$$

By Hölder inequality, we derive

$$\begin{aligned}
A_1 &\le \frac{1}{\triangle t}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0\left\|\mathbf{z}_h^{n+1}\right\|_0 \\
&\le \frac{1}{\beta\triangle t}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0\left\|e_h^{n+1}\right\|_0 \\
&\le \frac{C}{\beta^2\triangle t^2}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2 + \frac{1}{16}\left\|e_h^{n+1}\right\|_0^2,
\end{aligned} \tag{5.4.109}$$

and

$$\begin{aligned} A_2 &\leq \frac{C}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\nabla\mathbf{z}_h^{n+1}\right\|_0 \\ &\leq \frac{C}{\beta^2 Re^2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{1}{16}\left\|e_h^{n+1}\right\|_0^2. \end{aligned} \tag{5.4.110}$$

By Lemma 5.6, we have

$$\begin{aligned} A_3 &\leq \frac{C}{Re}\left\|s_h^{n+1} - s_h^n\right\|_0\left\|\mathrm{div}\,\mathbf{z}_h^{n+1}\right\|_0 \\ &\leq \frac{C}{\beta^2 Re^2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{1}{16}\left\|e_h^{n+1}\right\|_0^2, \end{aligned} \tag{5.4.111}$$

and

$$\begin{aligned} A_4 &\leq C\left\|\nabla\mathbf{z}_h^{n+1}\right\|_0\left\|\eta^{n+1}\right\|_0 \\ &\leq \frac{C}{\beta}\left\|e_h^{n+1}\right\|_0\left\|\eta^{n+1}\right\|_0 \\ &\leq \frac{1}{16}\left\|e_h^{n+1}\right\|_0^2 + \frac{C}{\beta^2}\left\|\eta^{n+1}\right\|_0^2. \end{aligned} \tag{5.4.112}$$

The last term in (5.4.108) can be split by

$$\begin{aligned} A_5 &= -\mathcal{N}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{z}_h^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{u}(t_n) - \mathbf{u}_h^n, \mathbf{u}(t_{n+1}), \mathbf{z}_h^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{u}_h^n - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \mathbf{z}_h^{n+1}) \\ &\quad -\mathcal{N}(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \mathbf{z}_h^{n+1}) \\ &= A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4}. \end{aligned} \tag{5.4.113}$$

By Lemma 1.16 and $\|\mathbf{u}(t_{n+1})\|_2 \leq C$, we get

$$\begin{aligned} A_{5,1} &\leq C\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\mathbf{z}_h^{n+1}\right\|_1 \\ &\leq \frac{C\triangle t}{\beta^2}\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + \frac{1}{16}\left\|e_h^{n+1}\right\|_0^2, \end{aligned} \tag{5.4.114}$$

and

$$
\begin{aligned}
A_{5,2} &\leq C\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_0\|\mathbf{u}(t_{n+1})\|_2\left\|\mathbf{z}_h^{n+1}\right\|_1 \\
&\leq \frac{C}{\beta^2}\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2\right)+\frac{1}{16}\left\|e_h^{n+1}\right\|_0^2.
\end{aligned} \tag{5.4.115}
$$

Since $\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)} \leq Ch^{-\frac{d}{6}}(\triangle t^{\frac{1}{2}}+h) \leq C_3$, we deduce

$$
\begin{aligned}
A_{5,3} &\leq \|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)}\left\|\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1}\right\|_1\left\|\mathbf{z}_h^{n+1}\right\|_1 \\
&\leq \frac{C}{\beta^2}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)+\frac{1}{16}\left\|e_h^{n+1}\right\|_0^2.
\end{aligned} \tag{5.4.116}
$$

By $\|\mathbf{u}(t_{n+1})\|_2 \leq C$, we obtain

$$
\begin{aligned}
A_{5,4} &\leq C\|\mathbf{u}(t_n)\|_2\left\|\mathbf{u}(t_{n+1})-\widehat{\mathbf{u}}_h^{n+1}\right\|_0\left\|\mathbf{z}_h^{n+1}\right\|_1 \\
&\leq \frac{C}{\beta^2}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)+\frac{1}{16}\left\|e_h^{n+1}\right\|_0^2.
\end{aligned} \tag{5.4.117}
$$

Upon gathering (5.4.114)-(5.4.117), (5.4.113) becomes

$$
\begin{aligned}
A_5 &\leq \frac{1}{4}\left\|e_h^{n+1}\right\|_0^2+\frac{C}{\beta^2}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\|\mathbf{E}^n\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2+\|\mathbf{G}^n\|_0^2\right) \\
&\quad +\frac{C\triangle t}{\beta^2}\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned} \tag{5.4.118}
$$

Plugging (5.4.109)-(5.4.112) and (5.4.118) into (5.4.108) implies

$$
\begin{aligned}
\frac{1}{2}\left\|e_h^{n+1}\right\|_0^2 &\leq \frac{C}{\beta^2\triangle t^2}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{C}{\beta^2 Re^2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\
&\quad +\frac{C}{\beta^2}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\|\mathbf{E}^n\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2+\|\mathbf{G}^n\|_0^2\right) \\
&\quad +\frac{C}{\beta^2}\left\|\eta^{n+1}\right\|_0^2+C\triangle t\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned} \tag{5.4.119}
$$

By multiply $2\triangle t$ and summation for n from 0 to N,

$$\begin{aligned}\triangle t\sum_{n=0}^{N}\left\|e_h^{n+1}\right\|_0^2 \leq \frac{C}{\beta^2\triangle t}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 \\ +\frac{C\triangle t}{\beta^2}\sum_{n=0}^{N+1}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\mathbf{E}^n\right\|_0^2+\left\|\mathbf{G}^n\right\|_0^2+\left\|\eta^{n+1}\right\|_0^2\right) \\ +\frac{C\triangle t}{\beta^2 Re^2}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\frac{C\triangle t^2}{\beta^2}\int_{t_0}^{t_{N+1}}\left\|\mathbf{u}_t(t)\right\|_0^2 dt.\end{aligned} \tag{5.4.120}$$

Finally, we get (5.4.105) by Lemmas 5.4, 5.7, and 5.16. ∎

5.5 Numerical Experiments

In this section we present a number of numerical experiments with the Gauge-Uzawa method and comparison with the Chorin-Uzawa method. The results show a superior performance of Gauge-Uzawa

5.5.1 Example : Smooth Solution on Distorted Mesh (a) in Figure 1.2

In all mesh analysis Figures 5.5, 5.1, 5.7, 5.3, the solid and dashed lines are errors of Gauge-Uzawa method and Chorin-Uzawa method, respectively. And The combinations 3.1-3.4 are explained in page 20. In Figure 5.1 which is $\triangle t = h^2$ and linear approximations of velocity and pressure, we can see the pressure errors of both methods do not converge to 0. So we conclude that the Gauge-Uzawa and Chorin-Uzawa methods depend on inf-sup condition. In the combination $\triangle t = h$, Figures 5.7 and 5.3, we can see the errors of Gauge-Uzawa method are little bit smaller than those of Chorin-Uzawa. These differences due to the inconsistency (2.2.6) of the Chorin-Uzawa.

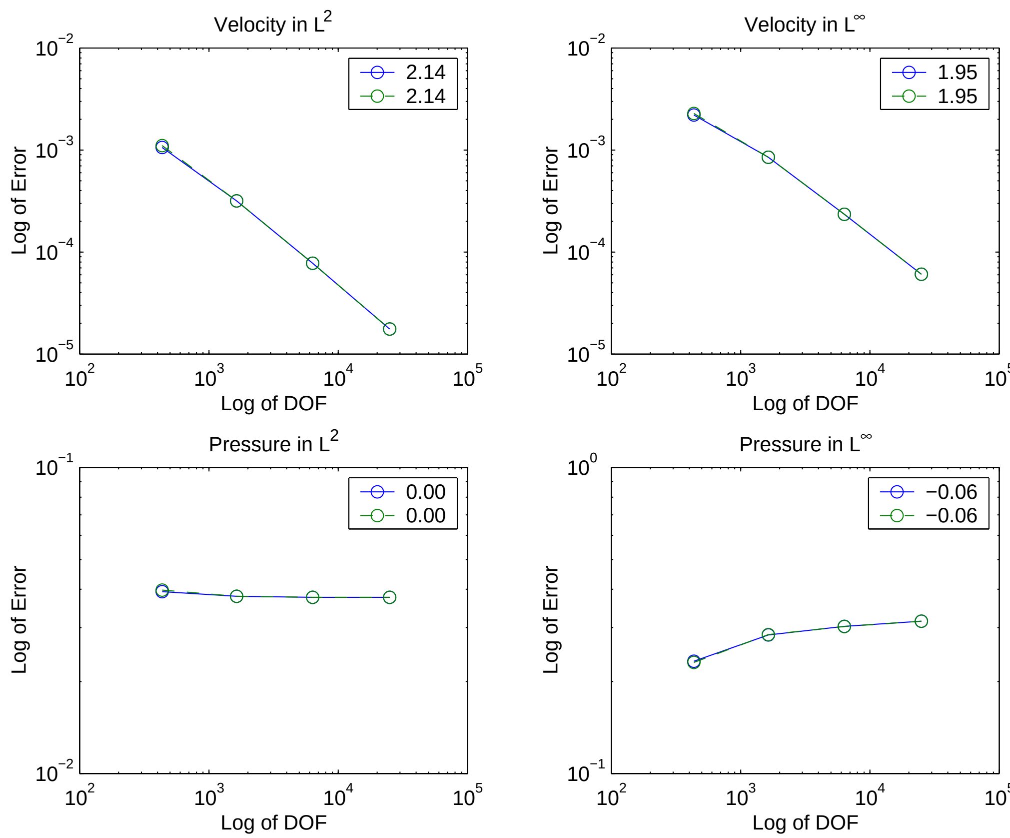


Figure 5.1: Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_1 - P_1$ Elements.

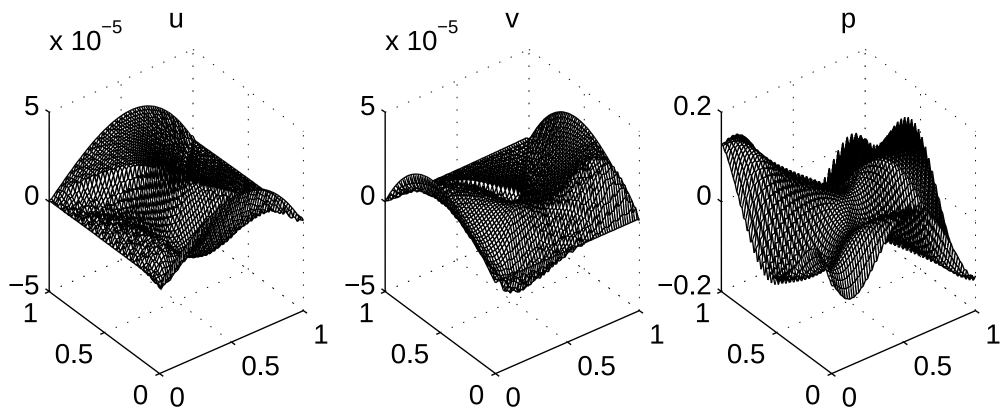


Figure 5.2: Error Functions for Gauge-Uzawa Method with $\triangle t = h^2$ and $P_1 - P_1$ Elements (DOF = 24,963).

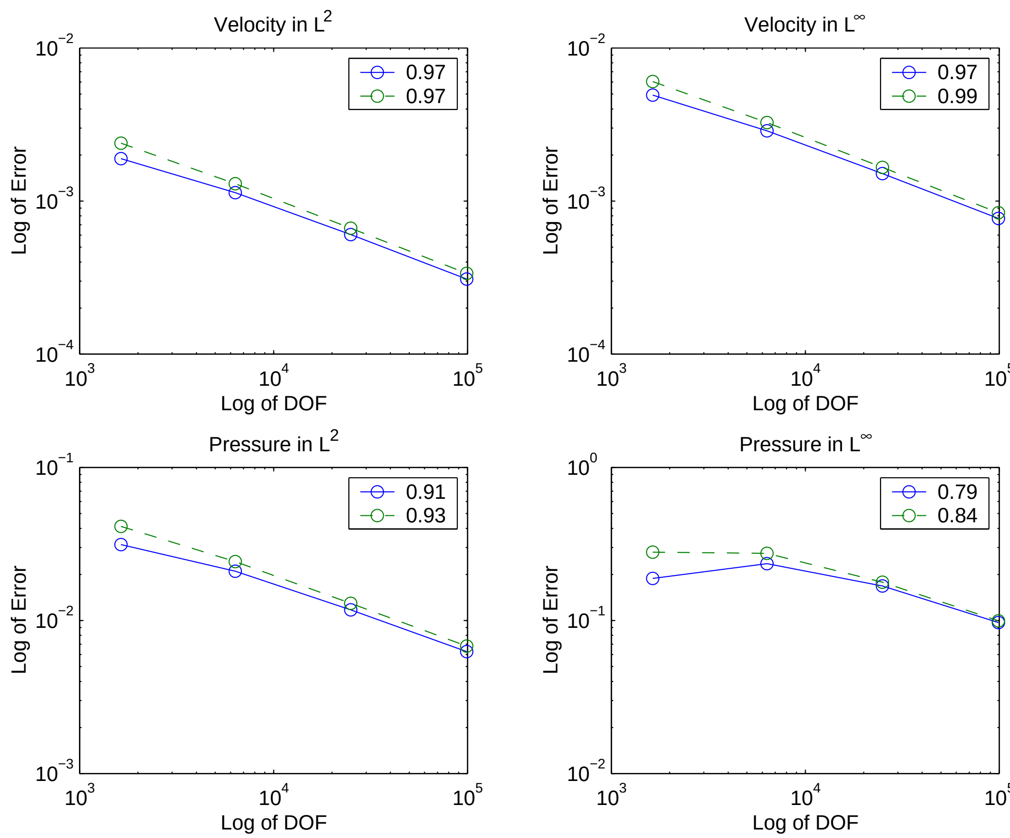


Figure 5.3: Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_1 - P_1$ Elements.

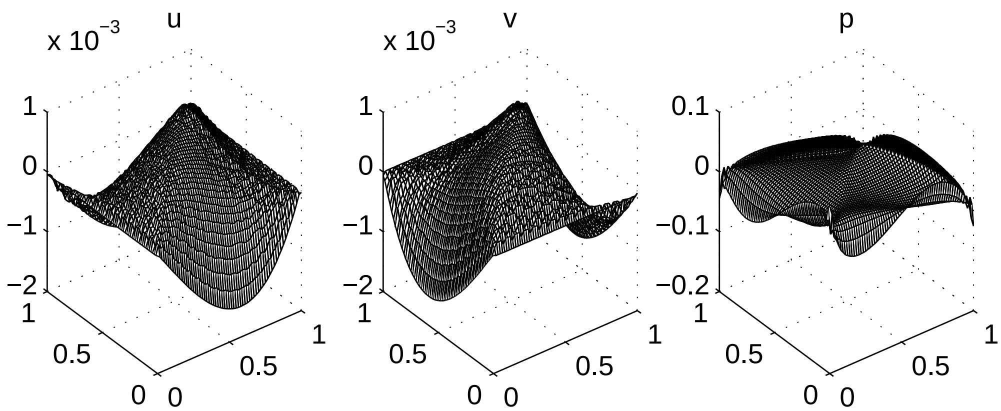


Figure 5.4: Error Functions for Gauge-Uzawa Method with $\triangle t = h$ and $P_1 - P_1$ Elements (DOF = 24,963).

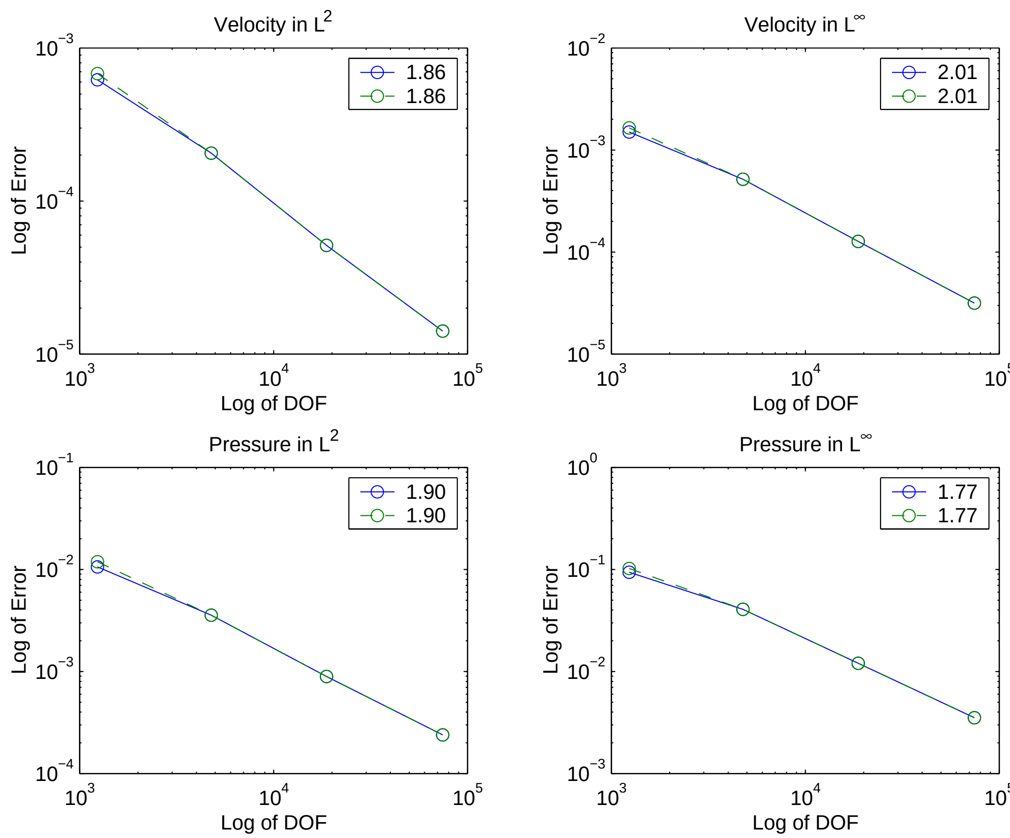


Figure 5.5: Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h^2$ and $P_2 - P_1$ Elements.

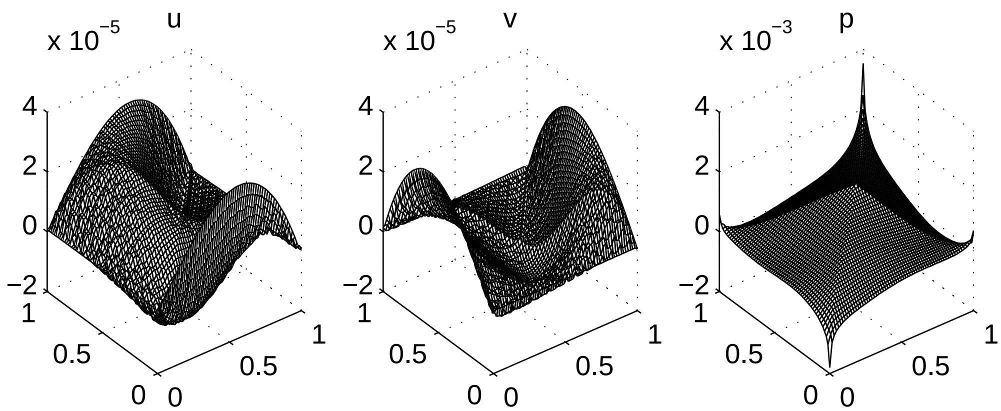


Figure 5.6: Error Functions for Gauge-Uzawa Method with $\triangle t = h^2$ and $P_2 - P_1$ Elements (DOF = 74,371).

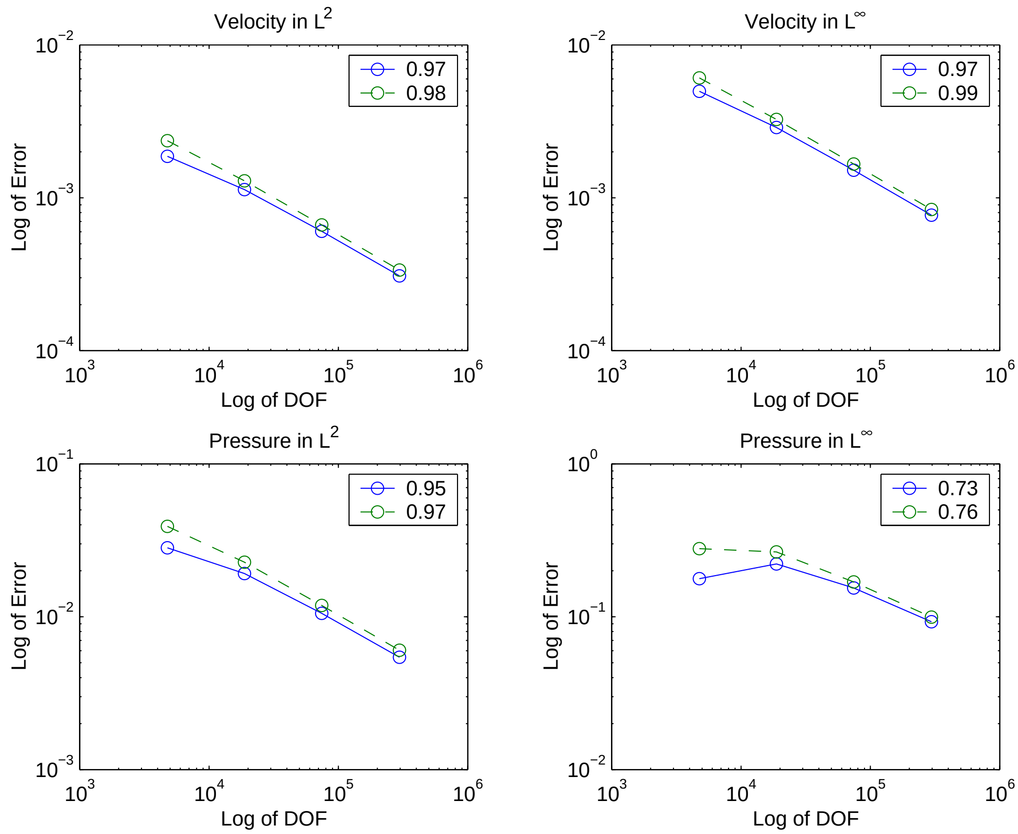


Figure 5.7: Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements.

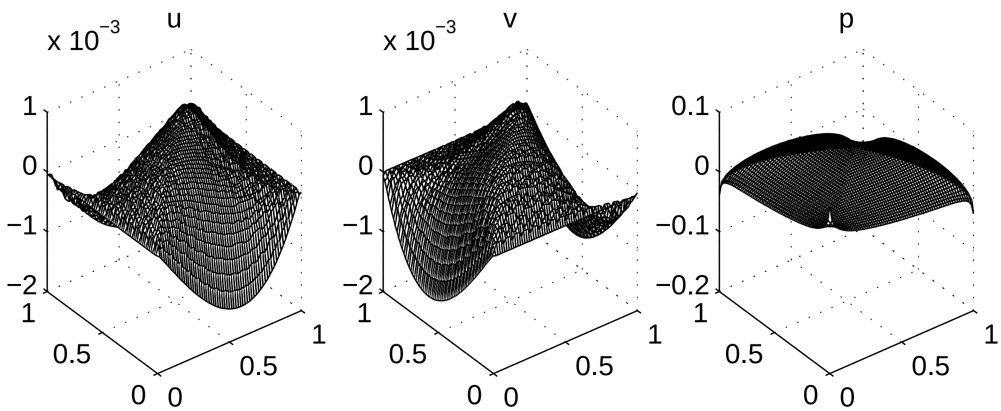


Figure 5.8: Error Functions for Gauge-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 74,371).

5.5.2 Example : Smooth Solution on Regular Mesh (b) in Figure 1.2

As we saw in Figure 5.1, the error of pressure does not decay to 0 without discrete inf-sup condition in distorted mesh. Gauge method show the error of pressure converges to 0 in Figure 3.25, even though it is not decay in distorted domain. But Figure 5.9 displays that the error of pressure for Gauge-Uzawa does not decay, like Chorin-Uzawa in Figure 2.13. Thus we conclude Gauge-Uzawa and Chorin-Uzawa methods are sensitive to discrete inf-sup condition even in regular mesh, but gauge method does not.

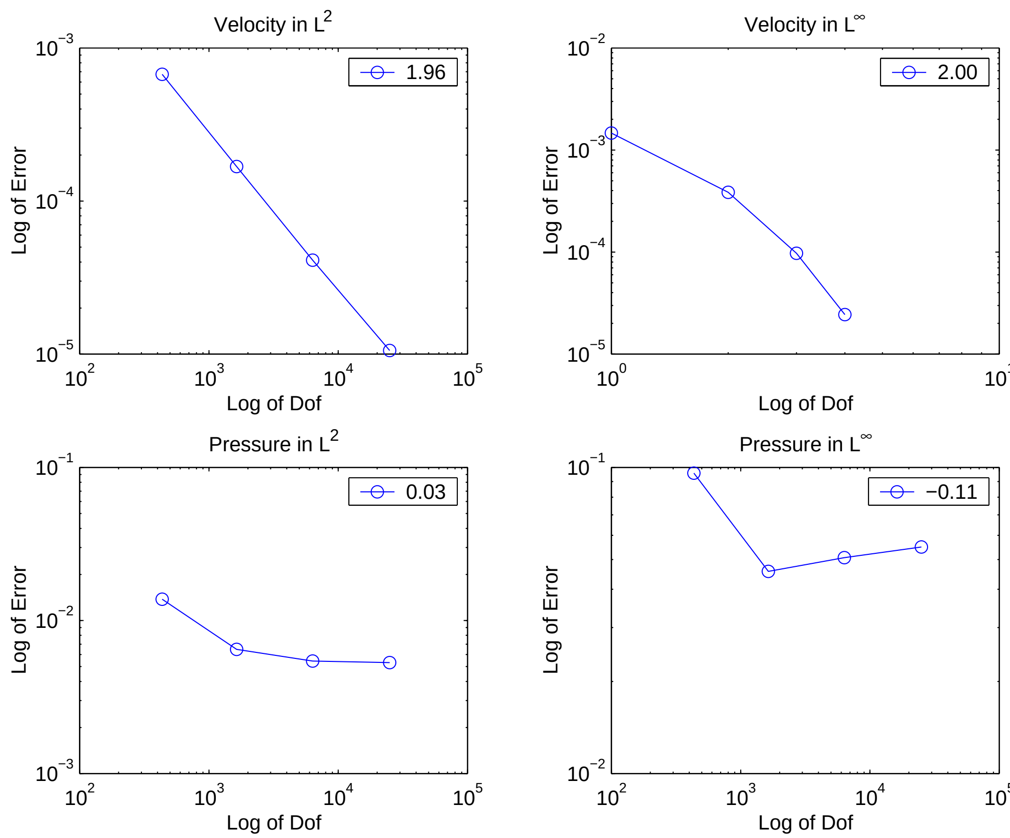


Figure 5.9: Error Decay of Gauge-Uzawa with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2.

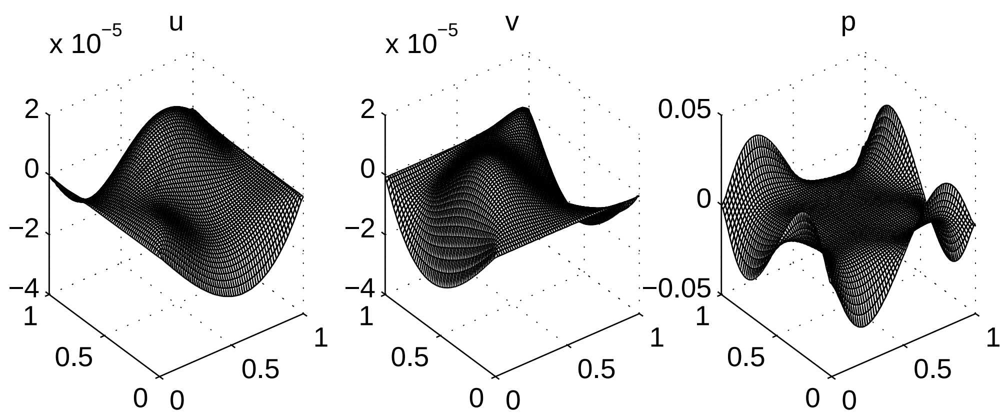


Figure 5.10: Error Functions of Gauge-Uzawa with $\triangle t = h^2$ and $P_1 - P_1$ Elements on Regular Mesh (b) in Figure 1.2 (DOF = 24,963).

5.5.3 Example : Singular Solution

We perform in this subsection Example 1.3.2 including singularity for pressure at the reentrant corner. As we studied in subsections 2.3.4 and 3.7.4, the other methods are suffering for the corner singularity, and give us not reliable results for this example. However the error of Gauge-Uzawa method is decreasing with same rate in Figure 5.11, also the pick is disappear for velocity at the reentrant corner in Figure 5.12. This example is a counter punch to make us conclude that the Gauge-Uzawa method is the higher level scheme than any other projection type methods.

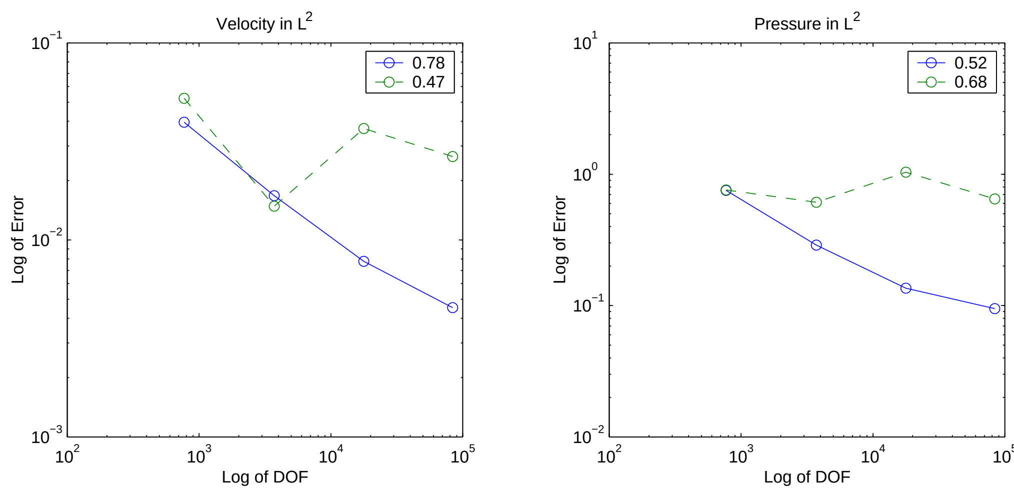


Figure 5.11: Error Decay of Gauge-Uzawa Method (Solid) and Chorin-Uzawa Method (Dashed) with $\triangle t = h$ and $P_2 - P_1$ Elements.

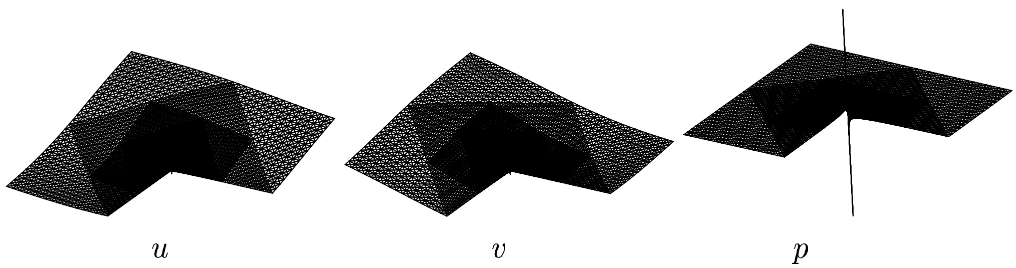


Figure 5.12: Numerical Solution of Gauge-Uzawa Method with $\triangle t = h$ and $P_2 - P_1$ Elements (DOF = 83,903).

5.5.4 Example : Forward Facing Step

This example is to find steady state solution for given initial value. A crucial difficulty of this example is that the initial condition is not satisfied divergence free. The domain $\Omega = ((-1,1)\times(-1,1)) - ((0,1]\times[-1,0])$ and the initial values

$$\begin{cases} u = 1.0 - y*y, & \text{if } x = -1, \\ u = 8.0*(1.0-y)y, & \text{if } x = 1, \\ u = 0, & \text{otherwise }, \end{cases} \tag{5.5.1}$$

where u is the first component of velocity, and the second component v is 0 in $\bar{\Omega}$. The non-dimensional numbers are chosen to be

$$Re = 200, \quad \triangle t = 0.01, \quad h = \tfrac{1}{2^5}. \tag{5.5.2}$$

The Figure 5.13 is n stationary flow in re-entrance domain. In this example, Re

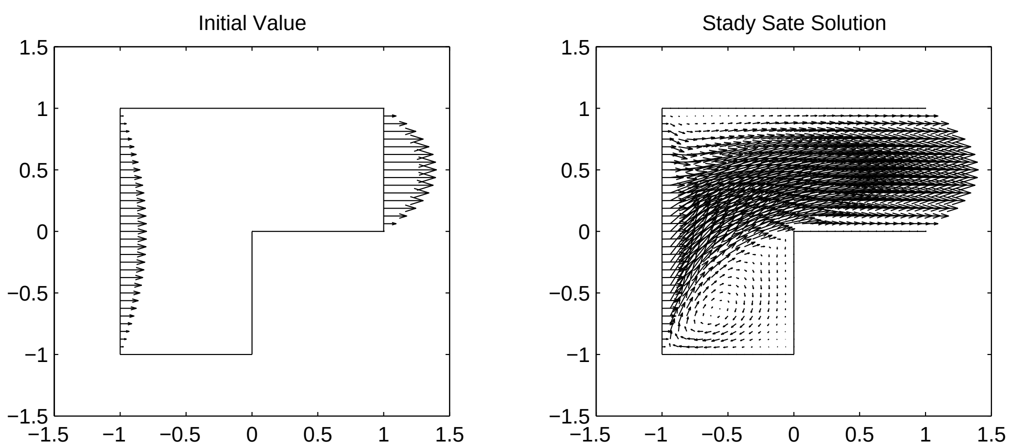


Figure 5.13: Initial and Steady State Solutions.

is 200, $\triangle t = 0.01$, and $h = \frac{1}{2^5}$.

5.5.5 Example : Driven Cavity Flow

This experiment is performed to find the steady state driven cavity flow with initial value 1 on top boundary for the first component u for velocity, and 0 on the others. The non-dimensional numbers are chosen to be

$$Re = 10,000, \quad \triangle t = 1, \quad h = \tfrac{1}{2^7}. \tag{5.5.3}$$

These extreme values are chosen to test the stability of the Gauge-Uzawa scheme which is proved in Section in 5.2.

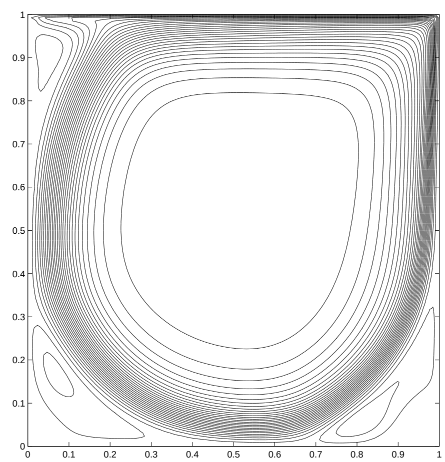


Figure 5.14: Driven Cavity Flow for $h = \frac{1}{128}$, $\triangle t = 1$, $Re = 10,000$.

Chapter 6

Gauge-Uzawa Method for the Evolution Boussinesq Equations

The goal of this chapter is to apply the Gauge-Uzawa method of Chapter 5 to the Boussinesq equations which govern the dynamics of incompressible fluids due to thermal driven convection. First, we introduce the following Boussinesq equations:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mu\triangle\mathbf{u} - \mathbf{g}\alpha\theta + \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \\ \theta_t + (\mathbf{u}\cdot\nabla)\theta = \beta\triangle\theta + b, & \text{in } \Omega, \\ \theta(\mathbf{x},0) = \theta_0, & \text{in } \Omega, \\ \theta = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.0.1}$$

where Ω is an open bounded convex polygon in $\mathbb{R}^d$ (d=2 or 3), $\mathbf{f}$ and b are forcing terms, $\mathbf{g}$ is the vector of gravitational acceleration, μ viscosity, α the coefficient of thermal expansion, and β the thermal diffusivity. The physically unknowns are velocity $\mathbf{u}$, pressure p, and temperature θ. Let L be a characteristic length, U a characteristic velocity, and Θ a characteristic temperature. Let the characteristic time T satisfy $T = \dfrac{L}{U}$. We can now measure $\mathbf{x}$, $\mathbf{u}$, θ, and t relative those scales by changing variables and introducing the following dimensionless quantities:

$$\widetilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \quad \widetilde{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \widetilde{\theta} = \frac{\theta}{\Theta}, \quad \widetilde{t} = \frac{t}{T}. \tag{6.0.2}$$

The Boussinesq equations can thus be written in the dimensionless form

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - \dfrac{1}{Re}\triangle\mathbf{u} = -\dfrac{Gr}{Re^2}\mathbf{g}\theta + \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \\ \theta_t + (\mathbf{u}\cdot\nabla)\theta = \dfrac{1}{RePr}\triangle\theta + b, & \text{in } \Omega, \\ \theta(\mathbf{x},0) = \theta_0, & \text{in } \Omega, \\ \theta = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.0.3}$$

where the non-dimensional numbers are defined as follows:

$$\begin{aligned} &\text{Reynolds number:} && Re = \frac{LU}{\mu} \\ &\text{Prandtl number:} && Pr = \frac{\mu}{\beta} \\ &\text{Grashof number:} && Gr = \frac{\alpha\Theta L^3}{\mu^2} \\ &\text{Rayleigh number:} && Ra = \frac{\mathbf{g}\alpha\Theta L^3}{\mu\beta} = GrPr \\ &\text{Richardson number:} && Ri = \frac{Gr}{Re^2}. \end{aligned} \tag{6.0.4}$$

6.1 Gauge-Uzawa Method for Boussinesq Equations

Now we apply Gauge-Uzawa method to solve Boussinesq equations. We use discrete velocity $\mathbb{V}_h$ and pressure $\mathbb{P}_h$ spaces as defined (1.2.38), and discrete temperature space is

$$\mathbb{T}_h = \{\theta_h \in H_0^1 : \theta_h|_K \in \mathcal{P}(K), \quad \forall K \in \mathfrak{T}\}. \tag{6.1.1}$$

We note the polynomial degrees of velocity and temperature spaces are the same; in both cases, the degree k of $\mathcal{P}(K)$ is $k \geq 1$ fixed.

Assumption 8 (Discrete Initial Condition for Temperature) *The initial θ_h^0 satisfies*

$$\left\langle \theta_h^0 \,,\, \psi_h \right\rangle = \left\langle \theta(t_0) \,,\, \psi_h \right\rangle, \quad \forall \psi_h \in \mathbb{T}_h. \tag{6.1.2}$$

Now, we introduce the Gauge-Uzawa method for the Boussinesq equations (6.0.3)

Algorithm 6.1 (Gauge-Uzawa Method for Boussinesq Equations) *Start with initial values $s_h^0 = 0$, $\mathbf{u}_h^0$ given in Assumption 7 and θ_h^0 given in Assumption 8,*

Step 1: *Compute $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$ as the solution of*

$$\begin{aligned} &\frac{\left\langle \widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n \,,\, \mathbf{w}_h \right\rangle}{\triangle t} + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \frac{1}{Re}\left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle \\ &-\frac{1}{Re}\left\langle s_h^n \,,\, \operatorname{div} \mathbf{w}_h \right\rangle = -\frac{Gr}{Re^2}\left\langle \mathbf{g}\theta^n \,,\, \mathbf{w}_h \right\rangle + \left\langle \mathbf{f}(t_{n+1}) \,,\, \mathbf{w}_h \right\rangle, \qquad \forall \mathbf{w}_h \in \mathbb{V}_h, \end{aligned} \tag{6.1.3}$$

Step 2: *Compute $\rho_h^{n+1} \in \mathbb{P}_h$ as the solution of*

$$\left\langle \nabla \rho_h^{n+1} \,,\, \nabla \psi_h \right\rangle = \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \,,\, \psi_h \right\rangle, \quad \forall \psi_h \in \mathbb{P}_h, \tag{6.1.4}$$

Step 3: *Update* $s_h^{n+1} \in \mathbb{P}_h$ *according to*

$$\left\langle s_h^{n+1}\,,\,q_h\right\rangle = \left\langle s^n\,,\,q_h\right\rangle - \left\langle \operatorname{div}\widehat{\mathbf{u}}_h^{n+1}\,,\,q_h\right\rangle, \quad \forall q_h \in \mathbb{P}_h. \tag{6.1.5}$$

Step 4: *Update* $\mathbf{u}_h^{n+1}$ *according to*

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla\rho_h^{n+1} \tag{6.1.6}$$

Step 5: *Compute* θ_h^{n+1} *as the solution of*

$$\begin{aligned} &\frac{\left\langle \theta_h^{n+1} - \theta_h^n\,,\,\psi_h\right\rangle}{\triangle t} + \mathcal{N}\left(\mathbf{u}_h^{n+1}, \nabla\theta_h^{n+1}, \psi_h\right) \\ &+ \frac{1}{RePr}\left\langle \nabla\theta_h^{n+1}\,,\,\nabla\psi_h\right\rangle = \left\langle b(t_{n+1})\,,\,\psi_h\right\rangle, \quad \forall\psi_h \in \mathbb{T}_h. \end{aligned} \tag{6.1.7}$$

If necessary, compute $p_h^{n+1} \in \mathbb{P}_h$ *according to*

$$p_h^{n+1} = -\frac{\rho_h^{n+1}}{\triangle t} + \frac{1}{Re}s_h^{n+1}. \tag{6.1.8}$$

6.2 Regularity of Boussinesq Equations

To prove stability and convergence of Algorithm 6.1 to the Boussinesq equations (6.0.3), it is enough to consider the temperature terms, because we have already examined those of the Navier-Stokes equations in Chapter 5. The goal of this section is to prove

$$\theta, \theta_t \in L^\infty\left(0, T; L^2(\Omega)\right) \in L^\infty(0, T; L^2(\Omega)), \tag{6.2.1}$$

which is the same regularity with forcing term $\mathbf{f}$ as in Assumption 2, in order to prove that Lemma 1.5 is still valid for the Boussinesq equations (6.0.3), and find realistic regularity results for temperature.

Assumption 9 *(Regularity for Temperature)*

$$\theta(0) \in H^2(\Omega) \text{ and } b,\ b_t \in L^\infty(0,\infty; L^2(\Omega)).$$

The following L^2-based a priori estimate for convex domains is well known in elliptic regularity theory [19].

Lemma 6.1 *Let Ω be a convex bounded domain. Let θ be the solution of*

$$\begin{cases} -\triangle\theta = f, & \text{in } \Omega \\ \theta = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.2.2}$$

Then we have the a priori estimate

$$\|\theta\|_2 \leq C\|f\|_0. \tag{6.2.3}$$

So we can use

$$\|\theta\|_2 \leq C\|\triangle\theta\|_0. \tag{6.2.4}$$

Lemma 6.2 *If Assumptions 1-3 and 9 hold, then the solution θ of the heat equation with convection in (6.0.3) satisfies*

$$\sup_{0\leq t\leq T}\left(\|\theta(t)\|_0 + \|\theta_t(t)\|_{-1} + \|\nabla\theta(t)\|_0\right) \leq M, \tag{6.2.5}$$

and

$$\frac{1}{RePr}\int_0^T \|\theta(t)\|_2^2 dt \leq M. \tag{6.2.6}$$

Proof. We note $\|\mathbf{u}\|_1 \leq C$ by Assumption 3. By multiplying by θ and the heat equation in (6.0.3), integrating, and integrating by parts, we get

$$\frac{d}{dt}\|\theta\|_0^2 + \frac{1}{RePr}\|\nabla\theta\|_0^2 \leq C\|b\|_{-1}^2. \tag{6.2.7}$$

Integrating both terms with respect to t derive

$$\|\theta(T)\|_0^2 + \frac{1}{RePr}\int_0^T \|\nabla\theta(t)\|_0^2 dt \le \|\theta(0)\|_0^2 + C\int_0^T \|b(t)\|_{-1}^2 dt \le M. \tag{6.2.8}$$

By multiplying heat equation in (6.0.3) by $-\triangle\theta$, integrating, (6.2.4), and Lemma 1.4, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_0^2 \quad &+ \frac{1}{RePr}\|\triangle\theta\|_0^2 \le \mathcal{N}(\mathbf{u},\theta,\triangle\theta) - \langle b\,,\,\triangle\theta\rangle \\ &\le \|\mathbf{u}\|_1\|\nabla\theta\|_1^{\frac{1}{2}}\|\nabla\theta\|_0^{\frac{1}{2}}\|\triangle\theta\|_0 + \|b\|_0\|\triangle\theta\|_0 \\ &\le C\|\triangle\theta\|_0^{\frac{3}{2}}\|\nabla\theta\|_0^{\frac{1}{2}} + \|b\|_0\|\triangle\theta\|_0 \\ &\le \frac{1}{2RePr}\|\triangle\theta\|_0^2 + CRe^3Pr^3\|\nabla\theta\|_0^2 + CRePr\|b\|_0^2.\end{aligned} \tag{6.2.9}$$

By integrating with respect to t, and by (6.2.8) and Assumption 9, we have

$$\begin{aligned}&\|\nabla\theta(T)\|_0^2 + \frac{1}{RePr}\int_0^T \|\triangle\theta(t)\|_0^2 dt \\ &\le \|\nabla\theta(0)\|_0^2 + CRe^3Pr^3\int_0^T \|\nabla\theta(t)\|_0^2 dt + CRePr\int_0^T \|b(t)\|_0^2 dt \le M.\end{aligned} \tag{6.2.10}$$

By using (6.2.8) and (6.2.10), we get

$$\begin{aligned}\|\theta_t\|_{-1} \quad &= \sup_{\eta\in H_0^1(\Omega)} \frac{\langle\theta_t\,,\,\eta\rangle}{\|\eta\|_1} \\ &= \sup_{\eta\in H_0^1(\Omega)} \frac{\left\langle -(\mathbf{u}\cdot\nabla)\theta + \frac{1}{RePr}\triangle\theta + b\,,\,\eta\right\rangle}{\|\eta\|_1} \\ &\le C\|\mathbf{u}\|_1\|\nabla\theta\|_0 + \frac{C}{RePr}\|\nabla\theta\|_0 + C\|b\|_{-1} \le M.\end{aligned} \tag{6.2.11}$$

The proof of Lemma 6.2 is completed. ■

Lemma 6.3 *If Assumptions 1-3 and 9 hold, then*

$$\sup_{0\le t\le T}\left(\|\mathbf{u}_t(t)\|_0 + \|\theta_t(t)\|_0\right) \le M \tag{6.2.12}$$

and

$$\int_0^T \left(\|\mathbf{u}(t)\|_2^2 + \|\nabla \mathbf{u}_t(t)\|_0^2 + \|\nabla \theta_t(t)\|_0^2\right) dt \le M. \qquad (6.2.13)$$

PROOF. We define a new variable $\widetilde{\mathbf{u}}$ as a projection of $-\triangle \mathbf{u}$. Then

$$\begin{cases} -\triangle \mathbf{u} + \nabla q = \widetilde{\mathbf{u}}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \qquad (6.2.14)$$

Then Assumption 1 gives us

$$\|\mathbf{u}\|_2 + \|q\|_1 \le \|\widetilde{\mathbf{u}}\|_0. \qquad (6.2.15)$$

Multiplying the momentum equation in (6.0.3) by $\widetilde{\mathbf{u}}$ and integrating by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_0^2 + \frac{1}{Re}\|\widetilde{\mathbf{u}}\|_0^2 \\ &= \mathcal{N}(\mathbf{u}, \mathbf{u}, \widetilde{\mathbf{u}}) - \left\langle \frac{Gr}{Re^2}\mathbf{g}\theta \,,\, \widetilde{\mathbf{u}} \right\rangle - \langle \mathbf{f} \,,\, \widetilde{\mathbf{u}} \rangle \\ &\le C\|\mathbf{u}\|_1 \|\nabla \mathbf{u}\|_1^{\frac{1}{2}} \|\nabla \mathbf{u}\|_0^{\frac{1}{2}} \|\widetilde{\mathbf{u}}\|_0 + \frac{CGr}{Re^2}\|\theta\|_0 \|\widetilde{\mathbf{u}}\|_0 + C\|\mathbf{f}\|_0 \|\widetilde{\mathbf{u}}\|_0 \\ &\le C\|\nabla \mathbf{u}\|_0^{\frac{1}{2}} \|\widetilde{\mathbf{u}}\|_0^{\frac{3}{2}} + \frac{CGr^2}{Re^3}\|\theta\|_0^2 + CRe\|\mathbf{f}\|_0^2 + \frac{1}{4Re}\|\widetilde{\mathbf{u}}\|_0^2 \\ &\le CRe^3\|\nabla \mathbf{u}\|_0^2 + \frac{CGr^2}{Re^3}\|\theta\|_0^2 + CRe\|\mathbf{f}\|_0^2 + \frac{1}{2Re}\|\widetilde{\mathbf{u}}\|_0^2. \end{aligned} \qquad (6.2.16)$$

Integrating with respect to t, we get, by Lemma 6.2, and Assumption 2 and 3,

$$\begin{aligned} &\|\nabla \mathbf{u}(T)\|_0^2 + \frac{1}{Re}\int_0^T \|\mathbf{u}(t)\|_2^2 dt \\ &\le \|\nabla \mathbf{u}(0)\|_0^2 + C\int_0^T \left(Re^3\|\nabla \mathbf{u}(t)\|_0^2 + \frac{Gr^2}{Re^3}\|\theta(t)\|_0^2 + Re\|\mathbf{f}(t)\|_0^2\right) dt \\ &\le M. \end{aligned} \qquad (6.2.17)$$

Multiplying the momentum equation by $\mathbf{u}_t$ and integrating, we have

$$\begin{aligned}
\|\mathbf{u}_t\|_0^2 &\leq C\|\mathbf{u}\|_2\|\nabla\mathbf{u}\|_0\|\mathbf{u}_t\|_0 + \frac{C}{Re}\|\triangle\mathbf{u}\|_0\|\mathbf{u}_t\|_0 \\
&\quad + C\|\mathbf{f}\|_0\|\mathbf{u}_t\|_0 + \frac{C}{Re^2}\|\theta\|_0\|\mathbf{u}_t\|_0 \\
&\leq C\|\mathbf{u}\|_2^2 + \frac{C}{Re^2}\|\triangle\mathbf{u}\|_0^2 + C\|\mathbf{f}\|_0^2 + \frac{C}{Re^4}\|\theta\|_0^2 + \frac{1}{2}\|\mathbf{u}_t\|_0^2.
\end{aligned} \tag{6.2.18}$$

By integrating with respect to t, we get, by Lemma 6.2 and formula (6.2.17),

$$\int_0^T \|\mathbf{u}_t(t)\|_0^2 dt \leq C\int_0^T \left(\|\mathbf{u}(t)\|_2^2 + \|\mathbf{f}(t)\|_0^2 + \frac{1}{Re^4}\|\theta(t)\|_0^2\right) dt \leq M. \tag{6.2.19}$$

We note, by (6.2.18),

$$\begin{aligned}
\|\mathbf{u}_t(0)\|_0^2 &\leq C\|\mathbf{u}(0)\|_2^2 + \frac{C}{Re^2}\|\triangle\mathbf{u}(0)\|_0^2 + C\|\mathbf{f}(0)\|_0^2 + \frac{C}{Re^4}\|\theta(0)\|_0^2 \\
&\leq C.
\end{aligned} \tag{6.2.20}$$

Now, differentiating momentum equation with respect to t, we have

$$\mathbf{u}_{tt} + (\mathbf{u}_t\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + \nabla p_t - \frac{1}{Re}\triangle\mathbf{u}_t = \frac{Gr}{Re^2}\theta_t + \mathbf{f}_t. \tag{6.2.21}$$

Multiplying (6.2.21) by $\mathbf{u}_t$ and integrating, and using Lemma 1.4, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|_0^2 + \frac{1}{Re}\|\nabla\mathbf{u}_t\|_0^2 \\
&\leq C\left(\|\nabla\mathbf{u}\|_0\|\mathbf{u}_t\|_1^{\frac{1}{2}}\|\mathbf{u}_t\|_0^{\frac{1}{2}}\|\mathbf{u}_t\|_1 + \frac{Gr}{Re^2}\|\theta_t\|_{-1}\|\mathbf{u}_t\|_1 + \|\mathbf{f}_t\|_{-1}\|\mathbf{u}_t\|_1\right) \\
&\leq C\left(\|\mathbf{u}_t\|_1^{\frac{3}{2}}\|\mathbf{u}_t\|_0^{\frac{1}{2}} + \frac{Gr}{Re^2}\|\theta_t\|_{-1}\|\mathbf{u}_t\|_1 + \|\mathbf{f}_t\|_{-1}\|\mathbf{u}_t\|_1\right) \\
&\leq CRe^3\|\mathbf{u}_t\|_0^2 + \frac{CGr^2}{Re^3}\|\theta_t\|_{-1}^2 + CRe\|\mathbf{f}_t\|_{-1}^2 + \frac{1}{2Re}\|\nabla\mathbf{u}_t\|_1^2.
\end{aligned} \tag{6.2.22}$$

Integrating with respect to t, and Lemma 6.2 and formula (6.2.19)-(6.2.20) imply

$$\begin{aligned} &\|\mathbf{u}_t(T)\|_0^2 + \frac{1}{Re}\int_0^T \|\nabla \mathbf{u}_t(t)\|_0^2 dt \\ &\le \|\mathbf{u}_t(0)\|_0^2 + C\int_0^T \left(Re^3\|\mathbf{u}_t(t)\|_0^2 + \frac{Gr^2}{Re^3}\|\theta_t(t)\|_{-1}^2 + \|\mathbf{f}_t(t)\|_{-1}^2 \right) dt \qquad (6.2.23) \\ &\le M. \end{aligned}$$

Now, we begin to estimate heat equation. (6.0.3) yield

$$\begin{aligned} \|\theta_t(0)\| &\le \|\mathbf{u}(0)\|_0 \|\nabla\theta(0)\|_0 + \frac{1}{RePr}\|\triangle\theta(0)\|_0 + \|b(0)\|_0 \\ &\le C. \end{aligned} \qquad (6.2.24)$$

Differentiating heat equation with respect to t yields

$$\theta_{tt} + (\mathbf{u}_t\cdot\nabla)\theta + (\mathbf{u}\cdot\nabla)\theta_t - \frac{1}{RePr}\triangle\theta_t = b_t. \qquad (6.2.25)$$

Multiplying (6.2.25) by θ_t and integrating imply

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\theta_t\|_0^2 + \frac{1}{RePr}\|\nabla\theta_t\|_0^2 \\ &\le C\|\mathbf{u}_t\|_0\|\theta\|_2\|\theta_t\|_1 + C\|b_t\|_{-1}\|\theta_t\|_1 \qquad (6.2.26) \\ &\le CRePr\left(\|\theta\|_2^2 + \|b_t\|_{-1}^2\right) + \frac{1}{2RePr}\|\nabla\theta_t\|_0^2. \end{aligned}$$

We note $\|\mathbf{u}_t\|_0 \le C$ by (6.2.23), and $\|\theta_t(0)\|_0^2 \le C$ by (6.2.24). Integrating with respect to t, we get, by Lemma 6.2,

$$\begin{aligned} &\|\theta_t(T)\|_0^2 + \frac{1}{RePr}\int_0^T \|\nabla\theta_t(t)\|_0^2 dt \\ &\le \|\theta_t(0)\|_0^2 + CRePr\int_0^T \left(\|\nabla\theta(t)\|_1^2 + \|b_t(t)\|_{-1}^2\right) dt \le M. \end{aligned} \qquad (6.2.27)$$

The proof of Lemma 6.3 is complete. ■

Since we now know θ and $\theta_t \in L^\infty(0,T;L^2(\Omega))$, we can consider the temperature θ in the momentum of (6.0.3) as forcing term $\mathbf{f}$. Consequently, we can use the regularity results of Lemma 1.5 for the Navier-Stokes equations and apply the resulting regularity of $(\mathbf{u}, p)$ to derive further regularity of θ.

Lemma 6.4 *If Assumption 1-3 and 9 hold, then*

$$\sup_{0<t<T} \|\theta(t)\|_2 \leq M, \quad \sup_{0<t<T} \sigma(t)\|\theta_t(t)\|_1 \leq M, \tag{6.2.28}$$

$$\int_0^T \sigma(t) \left(\|\theta_t\|_2^2 + \|\theta_{tt}\|_0^2\right) dt \leq M, \tag{6.2.29}$$

and

$$\int_0^T \|\theta_{tt}\|_{-1}^2 dt \leq M. \tag{6.2.30}$$

PROOF. From the heat equation, we get, by Lemmas 6.2 and 6.3,

$$\frac{1}{RePr}\|\triangle\theta\|_0 \leq \|\theta_t\|_0 + \|\mathbf{u}\|_0\|\nabla\theta\|_0 + \|b\|_0 \leq M. \tag{6.2.31}$$

Multiplying (6.2.25) by $-\triangle\theta_t$, and integrating imply

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|_0^2 + \frac{1}{RePr}\|\triangle\theta_t\|_0^2 \\ &\leq C\|\mathbf{u}_t\|_1\|\theta\|_2\|\triangle\theta_t\|_0 + C\|\mathbf{u}\|_2\|\nabla\theta_t\|_0\|\triangle\theta_t\|_0 + C\|b_t\|_0\|\triangle\theta_t\|_0 \\ &\leq CRePr\left(\|\mathbf{u}_t\|_1^2 + \|\nabla\theta_t\|_0^2 + \|b_t\|_0^2\right) + \frac{1}{2RePr}\|\triangle\theta_t\|_0^2. \end{aligned} \tag{6.2.32}$$

Since the chain rule give us

$$\frac{d}{dt}\left(\sigma(t)\|\nabla\theta_t(t)\|_0^2\right) = \sigma'(t)\|\nabla\theta_t(t)\|_0^2 + \sigma(t)\frac{d}{dt}\|\nabla\theta_t(t)\|_0^2, \tag{6.2.33}$$

Multiplying (6.2.32) by $\sigma(t)$ deduce

$$\begin{aligned} &\frac{d}{dt}\left(\sigma(t)\|\nabla\theta_t\|_0^2\right) + \frac{1}{RePr}\|\triangle\theta_t\|_0^2 \\ &\leq \sigma'(t)\|\nabla\theta_t(t)\|_0^2 + CRePr\sigma(t)\left(\|\mathbf{u}_t\|_1^2 + \|\nabla\theta_t\|_0^2 + \|b_t\|_0^2\right). \end{aligned} \tag{6.2.34}$$

Integrating from 0 to T for time t, we obtain

$$\begin{aligned} &\sigma(T)\|\nabla\theta_t(T)\|_0^2 + \frac{1}{RePr}\int_0^T \sigma(t)\|\theta_t\|_2^2 dt \\ &\leq \int_0^T \|\nabla\theta_t(t)\|_0^2 + CRePr\int_0^T \sigma(t)\left(\|\mathbf{u}_t\|_1^2 + \|\nabla\theta_t\|_0^2 + \|b_t\|_0^2\right) dt. \end{aligned} \tag{6.2.35}$$

the formula (6.2.35) can be bounded by Lemma 6.3. From (6.2.25), we derive

$$\begin{aligned} \|\theta_{tt}\|_0 &\leq C\|\mathbf{u}_t\|_0\|\nabla\theta\|_0 + C\|\mathbf{u}\|_0\|\nabla\theta_t\|_0 + \frac{C}{RePr}\|\triangle\theta_t\|_0 + C\|b_t\|_0 \\ &\leq C\left(\|\mathbf{u}_t\|_0 + \|\nabla\theta_t\|_0 + \frac{1}{RePr}\|\triangle\theta_t\|_0 + \|b_t\|_0\right). \end{aligned} \tag{6.2.36}$$

Squaring, multiplying $\sigma(t)$, and integrating from 0 to T for time t, we get

$$\begin{aligned} \int_0^T \sigma(t)\|\theta_{tt}\|_0^2 dt &\leq C\int_0^T \sigma(t)\left(\|\mathbf{u}_t\|_0^2 + \|\nabla\theta_t\|_0^2 + \|b_t\|_0^2\right) dt \\ &\quad + \frac{C}{RePr}\int_0^T \sigma(t)\|\theta_t\|_2^2 dt. \end{aligned} \tag{6.2.37}$$

By the formula (6.2.35) imply that (6.2.37) is bounded. Now we prove (6.2.30). The formula (6.2.25) and Lemma 6.3, give us

$$\begin{aligned} \|\theta_{tt}\|_{-1} &\leq C\left(\|\mathbf{u}_t\|_0\|\theta\|_2 + \|\mathbf{u}\|_2\|\theta_t\|_0 + \frac{1}{RePr}\|\nabla\theta_t\|_0 + \|b_t\|_{-1}\right) \\ &\leq C\left(\|\mathbf{u}_t\|_0 + \|\theta_t\|_0 + \frac{1}{RePr}\|\nabla\theta_t\|_0 + \|b_t\|_{-1}\right). \end{aligned} \tag{6.2.38}$$

Finally squaring, and integrating from 0 to T for time t, we derive

$$\begin{aligned} \int_0^T \|\theta_{tt}\|_{-1} dt &\leq C\int_0^T \left(\|\mathbf{u}_t\|_0 + \|\theta_t\|_0 + \|b_t\|_{-1}\right) dt \\ &\quad + \frac{1}{RePr}\int_0^T \|\nabla\theta_t\|_0 dt \leq M. \qquad \blacksquare \end{aligned} \tag{6.2.39}$$

6.3 Stability

In this section, we show that the Gauge-Uzawa method is unconditionally stable for the Boussinesq equations.

Theorem 6.1 (Stability) *Gauge-Uzawa method is unconditionally stable in the sense that for all $\triangle t > 0$ the following priori bound holds:*

$$
\begin{aligned}
\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + & \left\|\theta_h^{N+1}\right\|_0^2 + \frac{\triangle t}{Re}\left\|s_h^{N+1}\right\|_0^2 + 2\sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 \\
+ & \sum_{n=0}^{N}\left(\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + \left\|\theta_h^{n+1} - \theta_h^n\right\|_0^2\right) \\
+ & \frac{\triangle t}{RePr}\sum_{n=0}^{N}\left\|\nabla\theta_h^{n+1}\right\|_0^2 + \frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\
\leq & \left\|\mathbf{u}_h^0\right\|_0^2 + \left\|\theta_h^0\right\|_0^2 + \frac{\triangle t}{Re}\left\|s_h^0\right\|_0^2 \\
& + C\triangle t\sum_{n=0}^{N}\left(\|\mathbf{f}(t_{n+1})\|_{-1}^2 + \|b(t_{n+1})\|_{-1}^2\right).
\end{aligned}
\tag{6.3.1}
$$

Proof. By choosing $\mathbf{w}_h = 2\triangle t\widehat{\mathbf{u}}_h^{n+1}$ in the momentum formula (6.1.3) in Algorithm 6.1, we get

$$
\begin{aligned}
& 2\left\langle\widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 2\triangle t\mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1}) \\
& + \frac{2\triangle t}{Re}\left\langle\nabla\widehat{\mathbf{u}}_h^{n+1}\,,\,\nabla\widehat{\mathbf{u}}_h^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\operatorname{div}\widehat{\mathbf{u}}_h^{n+1}\right\rangle \\
& \qquad = -\frac{Gr\triangle t}{Re^2}\left\langle\mathbf{g}\theta^n\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 2\triangle t\left\langle\mathbf{f}(t_{n+1})\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle
\end{aligned}
\tag{6.3.2}
$$

or equivalently

$$
\begin{aligned}
& \left\|\mathbf{u}_h^{n+1}\right\|_0^2 - \left\|\mathbf{u}_h^n\right\|_0^2 + \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\
& = \frac{2\triangle t}{Re}\left\langle s_h^n\,,\,\operatorname{div}\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 2\triangle t\left\langle\mathbf{f}(t_{n+1})\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle - \frac{Gr\triangle t}{Re^2}\left\langle\mathbf{g}\theta^n\,,\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle \\
& = A_1 + A_2 + A_3.
\end{aligned}
\tag{6.3.3}
$$

We already estimate A_1 and A_2 by (5.2.7) and (5.2.8), respectively. And A_3 is bounded by

$$\begin{aligned} A_3 &\le \frac{Gr\triangle t}{Re^2}\|\theta^n\|_0\|\widehat{\mathbf{u}}_h^{n+1}\|_0 \\ &\le \frac{Gr^2\triangle t}{Re^4}\|\theta^n\|_0^2 + C\triangle t\|\widehat{\mathbf{u}}_h^{n+1}\|_0^2. \end{aligned} \tag{6.3.4}$$

We thus have

$$\begin{aligned} &\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_0^2 + 2\|\nabla\rho_h^{n+1}\|_0^2 + \frac{\triangle t}{2Re}\|\nabla\widehat{\mathbf{u}}_h^{n+1}\|_0^2 \\ &+\frac{\triangle t}{Re}\left(\|s_h^{n+1}\|_0^2 - \|s_h^n\|_0^2\right) \\ &\le C\triangle t\|\mathbf{f}(t_{n+1})\|_{-1}^2 + \frac{Gr^2\triangle t}{Re^4}\|\theta^n\|_0^2 + C\triangle t\|\widehat{\mathbf{u}}_h^{n+1}\|_0^2. \end{aligned} \tag{6.3.5}$$

Choosing $\psi_h = 2\triangle t\theta_h^{n+1}$ in the equation (6.1.6) yields

$$\begin{aligned} &\|\theta_h^{n+1}\|_0^2 - \|\theta_h^n\|_0^2 + \|\theta_h^{n+1} - \theta_h^n\|_0^2 + \frac{2\triangle t}{RePr}\|\nabla\theta_h^{n+1}\|_0^2 \\ &\le C\triangle t\|b(t_{n+1})\|_{-1}^2 + \triangle t\|\nabla\theta_h^{n+1}\|_0^2 \end{aligned} \tag{6.3.6}$$

or equivalently

$$\begin{aligned} &\|\theta_h^{n+1}\|_0^2 - \|\theta_h^n\|_0^2 + \|\theta_h^{n+1} - \theta_h^n\|_0^2 + \frac{\triangle t}{RePr}\|\nabla\theta_h^{n+1}\|_0^2 \\ &\le C\triangle t\|b(t_{n+1})\|_{-1}^2. \end{aligned} \tag{6.3.7}$$

By adding (6.3.5) and (6.3.7), we obtain

$$\begin{aligned} &\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_0^2 + 2\|\nabla\rho_h^{n+1}\|_0^2 + \frac{\triangle t}{2Re}\|\nabla\widehat{\mathbf{u}}_h^{n+1}\|_0^2 \\ &\|\theta_h^{n+1}\|_0^2 - \|\theta_h^n\|_0^2 + \|\theta_h^{n+1} - \theta_h^n\|_0^2 + \frac{\triangle t}{RePr}\|\nabla\theta_h^{n+1}\|_0^2 \\ &+\frac{\triangle t}{Re}\left(\|s_h^{n+1}\|_0^2 - \|s_h^n\|_0^2\right) \\ &\le C\triangle t\left(\|\mathbf{f}(t_{n+1})\|_{-1}^2 + \|b(t_{n+1})\|_{-1}^2\right) + \frac{Gr^2\triangle t}{Re^4}\|\theta^n\|_0^2 + C\triangle t\|\widehat{\mathbf{u}}_h^{n+1}\|_0^2. \end{aligned} \tag{6.3.8}$$

Upon summation over n from 0 to N, we deduce

$$\begin{aligned}
&\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + \left\|\theta_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left(\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + \left\|\theta_h^{n+1} - \theta_h^n\right\|_0^2\right) \\
&+\frac{\triangle t}{Re}\left\|s_h^{N+1}\right\|_0^2 + \frac{\triangle t}{RePr}\sum_{n=0}^{N}\left\|\nabla\theta_h^{n+1}\right\|_0^2 + \frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\
&+2\sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 \le \left\|\mathbf{u}_h^0\right\|_0^2 + \left\|\theta_h^0\right\|_0^2 + \frac{\triangle t}{Re}\left\|s_h^0\right\|_0^2 + \frac{Gr^2\triangle t}{Re^4}\sum_{n=0}^{N}\left\|\theta^n\right\|_0^2 \\
&\quad +C\triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{f}(t_{n+1})\right\|_{-1}^2 + \left\|b(t_{n+1})\right\|_{-1}^2\right) + C\triangle t\sum_{n=0}^{N}\left\|\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2.
\end{aligned} \tag{6.3.9}$$

Using Gronwall inequality, we prove the asserted estimate (6.3.1). ■

6.4 Error Estimate for Velocity and Temperature

In this section, we prove the convergence of velocity of fully discretized Algorithm 6.1 for Boussinesq equations. Since we have just verified stability and convergence of Gauge-Uzawa method in Navier-Stokes, we add the analysis for temperature terms in the proofs of NSE in Chapter 5. Since we proved θ and $\theta_t \in L^\infty(0,T;L^2(\Omega))$, we can use all lemmas in Chapter 5 upon considering temperature θ in the momentum equation as a force function.

Let $(\mathbf{U}^{n+1}, P^{n+1}) \in \mathbf{H}_0^1\Omega \times L_0^2(\Omega)$ be a weak solution of the following time discrete Stokes equations including exact convection and temperature:

$$\begin{cases} \left\langle \dfrac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\triangle t}\,,\,\mathbf{w}\right\rangle + \dfrac{1}{Re}\left\langle \nabla \mathbf{U}^{n+1}\,,\,\nabla \mathbf{w}\right\rangle - \left\langle P^{n+1}\,,\,\operatorname{div}\mathbf{w}\right\rangle \\ = \left\langle \mathbf{f}(t_{n+1})\,,\,\mathbf{w}\right\rangle - \mathcal{N}\left(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{w}\right) - \dfrac{Gr}{Re^2}\left\langle \mathbf{g}\theta(t_{n+1})\,,\,\mathbf{w}\right\rangle, \quad \forall \mathbf{w}\in \mathbf{H}_0^1(\Omega), \\ \left\langle q\,,\,\operatorname{div}\mathbf{U}^{n+1}\right\rangle = 0, \qquad \forall q \in L_0^2(\Omega). \end{cases} \tag{6.4.1}$$

We recall notation of Chapter 5 for

$$\mathbf{G}^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{U}^{n+1} \quad \text{and} \quad g^{n+1} = p(t_{n+1}) - P^{n+1}. \tag{6.4.2}$$

Since we prove θ and $\theta_t \in L^2\left(0,T;L^2(\Omega)\right)$, according to Lemma 6.2, we can use get directly the following estimates from Lemma 5.2:

Lemma 6.5 *Let Assumptions 1-3 hold. Then we have*

$$\left\|\mathbf{G}^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\|_0^2 + \frac{\triangle t}{Re}\sum_{n=0}^{N}\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 \le C\triangle t^2, \tag{6.4.3}$$

and

$$\triangle t\sum_{n=0}^{N}\left\|g^{n+1}\right\|_0^2 \le C\triangle t. \tag{6.4.4}$$

Let $(\mathbf{U}_h^{n+1}, P_h^{n+1}) \in \mathbb{V}_h \times \mathbb{P}_h$ be a discrete solution of the following weak Stokes equations

$$\begin{cases} \left\langle \nabla \mathbf{U}_h^{n+1}\,,\,\nabla \mathbf{w}_h\right\rangle + \left\langle \nabla P_h^{n+1}\,,\,\mathbf{w}_h\right\rangle \\ \qquad = \left\langle \nabla \mathbf{u}(t_{n+1})\,,\,\nabla \mathbf{w}_h\right\rangle + \left\langle \nabla p(t_{n+1})\,,\,\mathbf{w}_h\right\rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \left\langle r_h\,,\,\operatorname{div}\mathbf{U}_h^{n+1}\right\rangle = 0, \qquad \forall r_h \in \mathbb{P}_h, \end{cases} \tag{6.4.5}$$

where $(\mathbf{u}(t_{n+1}),\, p(t_{n+1}))$ is the exact solution of Boussinesq equations (6.0.3) at the time step t_{n+1}. We now recall the define error functions introduced in Chapter

5

$$\begin{aligned} \mathbf{G}_h^{n+1} &= \mathbf{u}(t_{n+1}) - \mathbf{U}_h^{n+1}, \quad g_h^{n+1} = p(t_{n+1}) - P_h^{n+1}, \\ \mathbf{F}^{n+1} &= \mathbf{U}^{n+1} - \mathbf{U}_h^{n+1}, \qquad \eta^{n+1} = P^{n+1} - P_h^{n+1}, \end{aligned} \tag{6.4.6}$$

and note that the following lemma follows from Lemmas 5.3 and 5.4.

Lemma 6.6 *Let Assumptions 1, and 4-6 hold, and let the exact solution* $(\mathbf{u}(t_{n+1}), p(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$. *Then we have*

$$\left\|\mathbf{G}_h^{n+1}\right\|_0 + h\left\|\mathbf{G}_h^{n+1}\right\|_1 \leq Ch^{s+1}\left(\|\mathbf{u}(t_{n+1})\|_{s+1} + \|p(t_{n+1})\|_s\right), \tag{6.4.7}$$

$$\left\|\left|\mathbf{G}_h^{n+1}\right|\right\| = \left\|\mathbf{G}_h^{n+1}\right\|_{L^\infty(\Omega)} + \left\|\mathbf{G}_h^{n+1}\right\|_{L^3(\Omega)} \leq M, \tag{6.4.8}$$

$$\left\|g_h^{n+1}\right\|_0 \leq Ch^s\left(\|\mathbf{u}(t_{n+1})\|_{s+1} + \|p(t_{n+1})\|_s\right), \tag{6.4.9}$$

$$\left\|\mathbf{F}^{n+1}\right\|_0 \leq \left\|\mathbf{G}^{n+1}\right\|_0 + \left\|\mathbf{G}_h^{n+1}\right\|_0 \leq C(\triangle t + h^{s+1}), \tag{6.4.10}$$

$$\begin{aligned} \triangle t \sum_{n=0}^{N} \left\|\nabla \mathbf{F}^{n+1}\right\|_0^2 &\leq \triangle t \sum_{n=0}^{N}\left(\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla \mathbf{G}_h^{n+1}\right\|_0^2\right) \\ &\leq C(\triangle t^2 + h^{2s}), \end{aligned} \tag{6.4.11}$$

and

$$\triangle t \sum_{n=0}^{N} \left\|\eta^{n+1}\right\|_0^2 \leq \triangle t \sum_{n=0}^{N}\left(\left\|g^{n+1}\right\|_0^2 + \left\|g_h^{n+1}\right\|_0^2\right) \leq C(\triangle t + h^{2s}). \tag{6.4.12}$$

We recall the additional error functions of Chapter 5

$$\begin{aligned} \mathbf{E}^{n+1} &= \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}^{n+1} = \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e^{n+1} = P^{n+1} - p_h^{n+1}, \\ \mathbf{E}_h^{n+1} &= \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}_h^{n+1} = \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e_h^{n+1} = P_h^{n+1} - p_h^{n+1}. \end{aligned} \tag{6.4.13}$$

The following properties mimic those of Lemma 5.5:

Lemma 6.7 *(Properties of Error Functions) We have the relations among the error functions in (6.4.13):*

$$\langle \mathbf{E}^{n+1}, \nabla q_h \rangle = \left\langle \mathbf{E}_h^{n+1}, \nabla q_h \right\rangle = \langle \mathbf{F}^{n+1}, \nabla q_h \rangle = 0, \quad \forall q_h \in \mathbb{P}_h, \tag{6.4.14}$$

$$\begin{aligned} \widehat{\mathbf{E}}^{n+1} &= \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1} \\ &= \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1} + \nabla \rho_h^{n+1} = \mathbf{E}^{n+1} + \nabla \rho_h^{n+1}, \end{aligned} \tag{6.4.15}$$

$$\begin{aligned} \widehat{\mathbf{E}}_h^{n+1} &= \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1} \\ &= \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1} + \nabla \rho_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla \rho_h^{n+1}, \end{aligned} \tag{6.4.16}$$

and

$$\widehat{\mathbf{E}}^{n+1} = \mathbf{F}^{n+1} + \widehat{\mathbf{E}}_h^{n+1} \quad \textit{and} \quad \mathbf{E}^{n+1} = \mathbf{F}^{n+1} + \mathbf{E}_h^{n+1}. \tag{6.4.17}$$

The following lemma can be derived directly from Lemma 5.6,

Lemma 6.8 *Let $s_h \in \mathbb{P}_h$ be defined in Algorithm 6.1. Then we have*

$$\left\| s_h^{n+1} - s_h^n \right\|_0^2 \le \left\| \nabla \widehat{\mathbf{E}}^{n+1} \right\|_0^2. \tag{6.4.18}$$

We now introduce several functions related to temperature. Let $\theta(t_{n+1})$ be exact temperature at time step $n+1$ and $\theta_h(t_{n+1}) \in \mathbb{T}_h$ be the L^2-projection of $\theta(t_{n+1})$:

$$\langle \theta_h(t_{n+1}), \psi_h \rangle = \langle \theta(t_{n+1}), \psi_h \rangle, \quad \forall \psi_h \in \mathbb{T}_h. \tag{6.4.19}$$

We define the error functions of temperature as follows:

$$\varepsilon^{n+1} = \theta(t_{n+1}) - \theta_h^{n+1}, \quad \varepsilon_h^{n+1} = \theta_h(t_{n+1}) - \theta_h^{n+1}, \tag{6.4.20}$$

and

$$\delta^{n+1} = \theta(t_{n+1}) - \theta_h(t_{n+1}). \tag{6.4.21}$$

Then for the interpolation error δ^{n+1}, we have

$$\left\|\delta^{n+1}\right\|_0 + h\left\|\delta^{n+1}\right\|_1 \le Ch^{s+1}\|\theta(t_{n+1})\|_{s+1} \tag{6.4.22}$$

and

$$\left\|\left|\delta^{n+1}\right|\right\| = \left\|\delta^{n+1}\right\|_{L^\infty(\Omega)} + \left\|\delta^{n+1}\right\|_{L^3(\Omega)} \le M. \tag{6.4.23}$$

Our purpose in the following lemma is to show that $\mathbf{u}_h^{n+1}$, $\widehat{\mathbf{u}}_h^{n+1}$, and θ_h^{n+1} are order $\mathcal{O}(\triangle t^{\frac{1}{2}} + h^s)$ approximations to their exact counterparts. This result will be used to improve the error estimates to $\mathcal{O}(\triangle t + h^{s+1})$ in Lemma 6.11.

Lemma 6.9 *Let Assumptions 1-9 hold, and let the exact solution of (6.0.3) satisfy*

$$(\mathbf{u}(t_{n+1}), p(t_{n+1}), \theta(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega) \times H^{s+1}(\Omega). \tag{6.4.24}$$

If $h^2 \le C_1 \triangle t$ *with* $C_1 > 0$ *arbitrary, then we have*

$$\begin{aligned} &\left\|\mathbf{E}^{N+1}\right\|_0^2 + \left\|\varepsilon^{N+1}\right\|_0^2 + \frac{1}{2}\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \left\|\varepsilon^{n+1} - \varepsilon^n\right\|_0^2\right) \\ &+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 + \frac{\triangle t}{RePr}\sum_{n=0}^{N}\left\|\nabla\varepsilon^{n+1}\right\|_0^2 \\ &+\frac{\triangle t}{Re}\left\|s_h^{n+1}\right\|_0^2 \le C(\triangle t + h^{2s}). \end{aligned} \tag{6.4.25}$$

Proof. Upon subtracting (6.1.3) from (6.4.1), we get

$$\begin{aligned} &\left\langle \frac{\widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n}{\triangle t}, \mathbf{w}_h \right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}, \nabla\mathbf{w}_h \right\rangle \\ &= \left\langle P^{n+1}, \operatorname{div}\mathbf{w}_h\right\rangle - \frac{1}{Re}\left\langle s_h^n, \operatorname{div}\mathbf{w}_h\right\rangle - \frac{Gr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1}) - \theta_h^n), \mathbf{w}_h\right\rangle \\ &-\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w}_h) + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h), \qquad \forall \mathbf{w}_h \in \mathbb{V}_h. \end{aligned} \tag{6.4.26}$$

If choose $\mathbf{w}_h = 2\triangle t\widehat{\mathbf{E}}_h^{n+1} = 2\triangle t(\widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1})$, then (6.4.26) becomes

$$
\begin{aligned}
&2\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n\,,\, \widehat{\mathbf{E}}_h^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\, \nabla\widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&= 2\triangle t\left\langle P^{n+1}\,,\, \mathrm{div}\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle s_h^n\,,\, \mathrm{div}\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1}) + 2\triangle t\mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}) \\
&\quad -\frac{2\triangle tGr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1}) - \theta_h^n)\,,\, \widehat{\mathbf{E}}_h^{n+1}\right\rangle.
\end{aligned}
\tag{6.4.27}
$$

By formulas (5.3.31) and (5.3.32), (6.4.27) becomes

$$
\begin{aligned}
&\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + 2\left\|\nabla\rho_h^{n+1}\right\|_0^2 \\
&= 2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n\,,\, \mathbf{F}^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\, \nabla\mathbf{F}^{n+1}\right\rangle \\
&\quad +2\triangle t\left\langle P^{n+1}\,,\, \mathrm{div}\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle - \frac{2\triangle t}{Re}\left\langle s_h^n\,,\, \mathrm{div}\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1}) - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1})\right) \\
&\quad -\frac{2\triangle tGr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1}) - \theta_h^n)\,,\, \widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned}
\tag{6.4.28}
$$

Since the estimate for A_1-A_5 in (6.4.28) are the same as (5.3.34)-(5.3.42) in Chapter 5, we need to show just the following A_6

$$
\begin{aligned}
A_6 \;\leq\; &\frac{C\triangle tGr}{Re^2}\left(\|\varepsilon^n\|_0^2 + \left\|\mathbf{F}^{n+1}\right\|_0^2 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\
&+\frac{C\triangle t^2Gr}{Re^2}\int_{t_n}^{t_{n+1}} \|\theta_t(t)\|_0^2 dt.
\end{aligned}
\tag{6.4.29}
$$

Upon plugging (5.3.34)-(5.3.42) and (6.4.29) into (6.4.28), we derive

$$\begin{aligned}
&\left\|\mathbf{E}^{n+1}\right\|_0^2-\|\mathbf{E}^n\|_0^2+\frac{1}{2}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2\\
&+\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2-\|s_h^n\|_0^2\right)\leq C\left\|\mathbf{F}^{n+1}\right\|_0^2+\frac{C\triangle t^2 Gr}{Re^2}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2dt.\\
&\quad+CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2+\left\|\eta^{n+1}\right\|_0^2\right)\\
&\quad+C\triangle t^2\left(\|\mathbf{u}(t_{n+1})\|_2^2+\|\nabla p(t_{n+1})\|_0^2\right)+CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt\\
&\quad+\frac{C\triangle t}{Re}\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2+\frac{C\triangle tGr}{Re^2}\left(\|\varepsilon^n\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right).
\end{aligned}\tag{6.4.30}$$

We now estimate temperature equation. By virtue of the Taylor theorem for the heat equation, we get

$$\begin{aligned}
&\frac{\theta(t_{n+1})-\theta(t_n)}{\triangle t}+(\mathbf{u}(t_{n+1})\cdot\nabla)\theta(t_{n+1})-\frac{1}{RePr}\triangle\theta(t_{n+1})\\
&\qquad=b(t_{n+1})+R_\theta^{n+1},
\end{aligned}\tag{6.4.31}$$

where $R_\theta(t_{n+1})=\dfrac{1}{\triangle t}\displaystyle\int_{t_n}^{t_{n+1}}(t-t_n)\theta_{tt}(t)dt$. Then the weak formulation of (6.4.31) is

$$\begin{aligned}
&\frac{1}{\triangle t}\left\langle\theta(t_{n+1})-\theta(t_n)\,,\,\psi\right\rangle+\mathcal{N}(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\psi)\\
&+\frac{1}{RePr}\left\langle\nabla\theta(t_{n+1})\,,\,\nabla\psi\right\rangle=\left\langle b(t_{n+1})\,,\,\psi\right\rangle+\left\langle R_\theta^{n+1}\,,\,\psi\right\rangle,\quad\forall\psi\in H_0^1(\Omega).
\end{aligned}\tag{6.4.32}$$

By subtracting (6.1.7) in Algorithm 6.1 from (6.4.32),

$$\begin{aligned}
&\frac{\left\langle\varepsilon^{n+1}-\varepsilon^n\,,\,\psi_h\right\rangle}{\triangle t}+\frac{1}{RePr}\left\langle\nabla\varepsilon^{n+1}\,,\,\nabla\psi_h\right\rangle=\left\langle R_\theta^{n+1}\,,\,\psi_h\right\rangle\\
&-\mathcal{N}(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\psi_h)+\mathcal{N}(\mathbf{u}_h^{n+1},\theta_h^{n+1},\psi_h),\quad\forall\psi_h\in\mathbb{T}_h.
\end{aligned}\tag{6.4.33}$$

By choosing $\psi_h=2\triangle t\varepsilon_h^{n+1}$, we have

$$\begin{aligned}
&2\left\langle\varepsilon^{n+1}-\varepsilon^n\,,\,\varepsilon_h^{n+1}\right\rangle+\frac{2\triangle t}{RePr}\left\langle\nabla\varepsilon^{n+1}\,,\,\nabla\varepsilon_h^{n+1}\right\rangle=2\triangle t\left\langle R_\theta^{n+1}\,,\,\varepsilon_h^{n+1}\right\rangle\\
&\quad-2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\varepsilon_h^{n+1})+2\triangle t\mathcal{N}(\mathbf{u}_h^{n+1},\theta_h^{n+1},\varepsilon_h^{n+1}).
\end{aligned}\tag{6.4.34}$$

Then the left hand side terms become

$$2\left\langle \varepsilon^{n+1}-\varepsilon^{n}\,,\,\varepsilon_h^{n+1}\right\rangle \;=\; \left\|\varepsilon^{n+1}\right\|_0^2-\left\|\varepsilon^{n}\right\|_0^2+\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_0^2 \\ -2\left\langle \varepsilon^{n+1}-\varepsilon^{n}\,,\,\delta^{n+1}\right\rangle \tag{6.4.35}$$

and

$$\frac{2\triangle t}{RePr}\left\langle \nabla\varepsilon^{n+1}\,,\,\nabla\varepsilon_h^{n+1}\right\rangle \;=\; \frac{2\triangle t}{RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2 \\ -\frac{2\triangle t}{RePr}\left\langle \nabla\varepsilon^{n+1}\,,\,\nabla\delta^{n+1}\right\rangle. \tag{6.4.36}$$

So (6.4.34) becomes

$$\begin{aligned} &\left\|\varepsilon^{n+1}\right\|_0^2-\left\|\varepsilon^{n}\right\|_0^2+\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_0^2+\frac{2\triangle t}{RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2 \\ &=2\triangle t\left\langle R_\theta^{n+1}\,,\,\varepsilon_h^{n+1}\right\rangle+2\left\langle \varepsilon^{n+1}-\varepsilon^{n}\,,\,\delta^{n+1}\right\rangle+\frac{2\triangle t}{RePr}\left\langle \nabla\varepsilon^{n+1}\,,\,\nabla\delta^{n+1}\right\rangle \\ &\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\varepsilon_h^{n+1})-\mathcal{N}(\mathbf{u}_h^{n+1},\theta_h^{n+1},\varepsilon_h^{n+1})\right) \\ &=A_7+A_8+A_9+A_{10}. \end{aligned} \tag{6.4.37}$$

We now estimate the split terms in 6.4.37). The beginning 3 terms are derived easily as follows:

$$A_7 \;\le\; \frac{\triangle t}{8RePr}\left(\left\|\nabla\varepsilon^{n+1}\right\|_0^2+\left\|\nabla\delta^{n+1}\right\|_0^2\right) \\ +CRePr\triangle t^2\int_{t_n}^{t_{n+1}}\|\theta_{tt}(t)\|_{-1}^2dt, \tag{6.4.38}$$

$$A_8\le\frac{1}{2}\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_0^2+C\left\|\delta^{n+1}\right\|_0^2, \tag{6.4.39}$$

and

$$A_9\le\frac{\triangle t}{8RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2+\frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2. \tag{6.4.40}$$

The last term in (6.4.37) can be split by

$$\begin{aligned} A_{10} &= -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}, \theta(t_{n+1}), \varepsilon_h^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}_h^{n+1}, \delta^{n+1}, \varepsilon_h^{n+1}) - 2\triangle t\mathcal{N}(\mathbf{u}_h^{n+1}, \varepsilon_h^{n+1}, \varepsilon_h^{n+1}) \\ &= A_{10,1} + A_{10,2} + A_{10,3}. \end{aligned} \tag{6.4.41}$$

Here we note $A_{10,3} = 0$ by (1.2.61), and the others are estimated by

$$\begin{aligned} A_{10,1} &\leq C\triangle t\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\right\|_0 \|\theta(t_{n+1})\|_2 \left\|\nabla\varepsilon_h^{n+1}\right\|_0 \\ &\leq CRePr\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2\right) + \frac{\triangle t}{4RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2 \\ &\quad +\frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2, \end{aligned} \tag{6.4.42}$$

and

$$\begin{aligned} A_{10,2} &= 2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}, \delta^{n+1}, \varepsilon_h^{n+1}) \\ &\quad -2\triangle t\mathcal{N}(\mathbf{u}(t_{n+1}), \delta^{n+1}, \varepsilon_h^{n+1}) = A_{10,4} + A_{10,5}. \end{aligned} \tag{6.4.43}$$

Since we have $\|\delta^{n+1}\| \leq M$ by (6.4.23), Lemma 1.16 implies

$$\begin{aligned} A_{10,4} &\leq C\triangle t\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\right\|_0 \left\|\delta^{n+1}\right\| \left\|\nabla\varepsilon_h^{n+1}\right\|_0 \\ &\leq CRePr\triangle t\left(\left\|\mathbf{G}^{n+1}\right\|_0^2 + \left\|\mathbf{E}^{n+1}\right\|_0^2\right) \\ &\quad +\frac{\triangle t}{4RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0 + \frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2. \end{aligned} \tag{6.4.44}$$

By Lemma 1.4, we obtain

$$\begin{aligned} A_{10,5} &\leq C\triangle t\|\mathbf{u}(t_{n+1})\|_2 \left\|\delta^{n+1}\right\|_0 \left\|\nabla\varepsilon_h^{n+1}\right\|_0 \\ &\leq CRePr\triangle t\left\|\delta^{n+1}\right\|_0^2 + \frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2 + \frac{\triangle t}{4RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2. \end{aligned} \tag{6.4.45}$$

Upon plugging (6.4.38)-(6.4.45) into (6.4.37) we deduce

$$\begin{aligned}
&\left\|\varepsilon^{n+1}\right\|_0^2-\|\varepsilon^n\|_0^2+\frac{1}{2}\left\|\varepsilon^{n+1}-\varepsilon^n\right\|_0^2+\frac{\triangle t}{RePr}\|\nabla\varepsilon(t_{n+1})\|_0^2\\
&\le CRePr\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)+\frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2\\
&\quad+C\left\|\delta^{n+1}\right\|_0^2+CRePr\triangle t^2\int_{t_n}^{t_{n+1}}\|\theta_{tt}(t)\|_{-1}^2dt.
\end{aligned}\tag{6.4.46}$$

The adding (6.4.30) and (6.4.46) implies

$$\begin{aligned}
&\left\|\mathbf{E}^{n+1}\right\|_0^2-\|\mathbf{E}^n\|_0^2+\frac{1}{2}\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2+\frac{\triangle t}{2Re}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2\\
&+\left\|\varepsilon^{n+1}\right\|_0^2-\|\varepsilon^n\|_0^2+\frac{1}{2}\left\|\varepsilon^{n+1}-\varepsilon^n\right\|_0^2+\frac{\triangle t}{RePr}\left\|\nabla\varepsilon^{n+1}\right\|_0^2\\
&+\frac{\triangle t}{Re}\left(\left\|s_h^{n+1}\right\|_0^2-\|s_h^n\|_0^2\right)\le CRePr\triangle t\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)\\
&\quad+CRe\triangle t\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2+\left\|\eta^{n+1}\right\|_0^2\right)\\
&\quad+C\triangle t^2\left(\|\mathbf{u}(t_{n+1})\|_2^2+\|\nabla p(t_{n+1})\|_0^2\right)+\frac{C\triangle t}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2\\
&\quad+C\left\|\mathbf{F}^{n+1}\right\|_0^2+\frac{C\triangle t}{Re}\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2+CRePr\triangle t^2\int_{t_n}^{t_{n+1}}\|\theta_{tt}(t)\|_{-1}^2dt\\
&\quad+C\left\|\delta^{n+1}\right\|_0^2+\frac{C\triangle tGr}{Re^2}\left(\|\varepsilon^n\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right)\\
&\quad+CRe\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt+\frac{C\triangle t^2Gr}{Re^2}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2dt.
\end{aligned}\tag{6.4.47}$$

On adding over n from 0 to N,

$$
\begin{aligned}
&\left\|\mathbf{E}^{N+1}\right\|_0^2+\left\|\varepsilon^{N+1}\right\|_0^2+\frac{1}{2}\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2+\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_0^2\right)\\
&+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2+\frac{\triangle t}{RePr}\sum_{n=0}^{N}\left\|\nabla\varepsilon^{n+1}\right\|_0^2\\
&+\frac{\triangle t}{Re}\left\|s_h^{N+1}\right\|_0^2\leq\left\|\mathbf{E}^0\right\|_0^2+\left\|\varepsilon^0\right\|_0^2\\
&\quad+CRePr\triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)\\
&\quad+CRe\triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n}\right\|_0^2+\left\|\mathbf{G}^{n}\right\|_0^2+\left\|\mathbf{G}_h^{n+1}\right\|_0^2+\left\|\eta^{n+1}\right\|_0^2\right)\\
&\quad+\frac{C\triangle tGr}{Re^2}\sum_{n=0}^{N}\left(\left\|\varepsilon^{n}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right)\\
&\quad+C\triangle t^2\sum_{n=0}^{N}\left(\left\|\mathbf{u}(t_{n+1})\right\|_2^2+\left\|\nabla p(t_{n+1})\right\|_0^2\right)+\frac{C\triangle t}{RePr}\sum_{n=0}^{N}\left\|\nabla\delta^{n+1}\right\|_0^2\\
&\quad+C\sum_{n=0}^{N}\left(\left\|\delta^{n+1}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2\right)+\frac{C\triangle t^2Gr}{Re^2}\int_{t_0}^{t_{N+1}}\left\|\theta_t(t)\right\|_0^2dt\\
&\quad+CRePr\triangle t\int_{t_0}^{t_{N+1}}\left\|\theta_{tt}(t)\right\|_{-1}^2dt+CRe\triangle t^2\int_{t_0}^{t_{N+1}}\left\|\mathbf{u}_t(t)\right\|_0^2dt.
\end{aligned}
\tag{6.4.48}
$$

We note that assumption $h^2\leq C_1\triangle t$, in conjunction with (6.4.10) and (6.4.22), yields

$$
C\sum_{n=0}^{N}\left(\left\|\delta^{n+1}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2\right)\leq C(\triangle t+h^{2s}). \tag{6.4.49}
$$

Since $\|\mathbf{E}^0\|_0^2+\|\varepsilon^0\|_0^2\leq Ch^{2s+2}\left(\|\mathbf{u}(0)\|_{s+1}^2+\|\theta(0)\|_{s+1}^2\right)$, the discrete Gronwall lemma, together with Lemmas 6.5-6.6, and (6.4.22), implies (6.4.25). ■

Let $(\mathbf{v}^{n+1},q^{n+1})\in\mathbf{H}^{s+1}(\Omega)\times H^s(\Omega)$ be the solution of the Stokes equations (5.3.45), and let $(\mathbf{v}_h^{n+1},q_h^{n+1})\in\mathbb{V}_h\times\mathbb{P}_h$ be a discrete solution of (5.3.46).

Let $\chi^{n+1} \in H_1^0(\Omega)$ be the solution of the Poisson equation

$$\begin{cases} -\triangle\chi^{n+1} = \varepsilon^{n+1}, \text{ in } \Omega, \\ \chi^{n+1} = 0, \text{ on } \partial\Omega. \end{cases} \tag{6.4.50}$$

In view of Lemma 6.1, χ^{n+1} satisfies

$$\left\|\chi^{n+1}\right\|_2 \leq C\left\|\varepsilon^{n+1}\right\|_0. \tag{6.4.51}$$

Let $\chi_h^{n+1} \in \mathbb{T}_h$ be the discrete counterpart of χ, namely,

$$\left\langle \nabla\chi_h^{n+1}, \nabla\xi_h \right\rangle = \left\langle \varepsilon^{n+1}, \xi_h \right\rangle, \quad \forall \xi_h \in \mathbb{T}_h. \tag{6.4.52}$$

Such a χ_h satisfies the interpolation estimate [1, 12, 28]

$$\begin{aligned} \left\|\chi^{n+1} - \chi_h^{n+1}\right\|_0 + h\left\|\chi^{n+1} - \chi_h^{n+1}\right\|_1 &\leq Ch^2\|\chi^{n+1}\|_2 \\ &\leq Ch^2\|\varepsilon^{n+1}\|_0 \end{aligned} \tag{6.4.53}$$

whence, from Lemma 1.17,

$$\left\|\left|\chi^{n+1} - \chi_h^{n+1}\right|\right\|_0 \leq C\left\|\varepsilon^{n+1}\right\|_0. \tag{6.4.54}$$

We have the following basic lemma [12, 28]

Lemma 6.10 *Let ε^{n+1} and χ^{n+1} be the solutions of (6.4.50) and (6.4.52), respectively. Then there exist positive constants C_1, C_2, such that*

$$C_1\left\|\varepsilon^{n+1}\right\|_{-1} \leq \left\|\nabla\chi^{n+1}\right\|_0 \leq C_2\left\|\varepsilon^{n+1}\right\|_{-1}. \tag{6.4.55}$$

Then we can obtain easily

$$\left\|\varepsilon^{n+1}\right\|_{-1} \leq C\left(\left\|\nabla\chi_h^{n+1}\right\|_0 + h\left\|\varepsilon^{n+1}\right\|_0\right). \tag{6.4.56}$$

Lemma 6.11 *Let Assumptions 1-9 hold, and let the exact solutions of 6.0.3 satisfy*

$$(\mathbf{u}(t_{n+1}), p(t_{n+1}), \theta(t_{n+1})) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega) \times H^{s+1}(\Omega). \tag{6.4.57}$$

If $h^2 \leq C\triangle t$, then we have

$$\begin{aligned} &\left\|\mathbf{E}_h^{N+1}\right\|_{\mathbf{Z}^*}^2 + \left\|\varepsilon^{N+1}\right\|_{-1}^2 + \frac{\triangle t}{2Re}\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\ &+\frac{2\triangle t}{RePr}\sum_{n=0}^{N}\left\|\varepsilon^{n+1}\right\|_0^2 + \sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_{\mathbf{Z}^*}^2 + \left\|\varepsilon^{n+1} - \varepsilon^n\right\|_{-1}^2\right) \\ &\leq C\left(\triangle t^2 + h^{2s+2}\right). \end{aligned} \tag{6.4.58}$$

Proof. We choose $\mathbf{w}_h = 2\triangle t\mathbf{v}_h^{n+1}$ in formula (6.4.26) then we have

$$\begin{aligned} &2\left\langle\widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n\,,\,\mathbf{v}_h^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle\nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{v}_h^{n+1}\right\rangle \\ &= 2\triangle t\left\langle P^{n+1}\,,\,\mathrm{div}\;\mathbf{v}_h^{n+1}\right\rangle - \frac{2\triangle tGr}{Re^2}\left\langle\mathbf{g}(\theta(t_{n+1}) - \theta_h^n)\,,\,\mathbf{v}_h^{n+1}\right\rangle \\ &\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h^{n+1}\right) + 2\triangle t\mathcal{N}\left(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1}\right). \end{aligned} \tag{6.4.59}$$

We note $\left\langle s_h\,,\,\mathrm{div}\;\mathbf{v}_h^{n+1}\right\rangle = 0$. By (5.3.54) and (5.3.55), we obtain

$$\begin{aligned} &\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 - \left\|\nabla\mathbf{v}_h^n\right\|_0^2 + \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\ &= -\frac{2\triangle t}{Re}\left\langle\nabla\mathbf{F}^{n+1}\,,\,\nabla\mathbf{v}_h^{n+1}\right\rangle + \frac{2\triangle t}{Re}\left\langle\mathbf{F}^{n+1}\,,\,\mathbf{E}^{n+1}\right\rangle \\ &\quad +\frac{2\triangle t}{Re}\left\langle\nabla\rho_h^{n+1}\,,\,\nabla q_h^{n+1}\right\rangle + 2\triangle t\left\langle P^{n+1}\,,\,\mathrm{div}\;\mathbf{v}_h^{n+1}\right\rangle \\ &\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h^{n+1}) - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1})\right) \\ &\quad -\frac{2\triangle tGr}{Re^2}\left\langle\mathbf{g}(\theta(t_{n+1}) - \theta_h^n)\,,\,\mathbf{v}_h^{n+1}\right\rangle \\ &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \end{aligned} \tag{6.4.60}$$

Since A_1-A_5 are the same terms as (5.3.57)-(5.3.68), it is enough to estimate the following A_6,

$$\begin{aligned} A_6 &\leq \frac{C\triangle t Gr}{Re^2}\|\theta(t_{n+1})-\theta_h^n\|_0\left\|\mathbf{v}_h^{n+1}\right\|_0 \\ &\leq \frac{\triangle t}{2RePr}\left(\left\|\varepsilon^{n+1}\right\|_0^2+\left\|\varepsilon^{n+1}-\varepsilon^n\right\|_0^2\right)+\frac{CGr^2Pr\triangle t}{Re^3}\left\|\mathbf{v}_h^{n+1}\right\|_0^2 \\ &+\frac{C\triangle t^2}{RePr}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2dt. \end{aligned} \tag{6.4.61}$$

Upon plugging (5.3.57)-(5.3.68) and (6.4.61) into (6.4.60), we deduce

$$\begin{aligned} &\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2-\|\nabla\mathbf{v}_h^n\|_0^2+\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+\frac{\triangle t}{Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\ &\leq CRe\triangle t\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2+CRe\triangle t h^2\left\|\eta^{n+1}\right\|_0^2+C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2dt \\ &+\frac{C\triangle t}{Re}\left(h^2\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right. \\ &\left.+\left\|\mathbf{G}^{n+1}\right\|_0^2+\|\mathbf{G}^n\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2\right)+\frac{CGr^2Pr\triangle t}{Re^3}\left\|\mathbf{v}_h^{n+1}\right\|_0^2 \\ &+CRe\triangle t(\triangle t+h^{2s})\left(\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\ &+\frac{\triangle t}{2RePr}\left(\left\|\varepsilon^{n+1}\right\|_0^2+\left\|\varepsilon^{n+1}-\varepsilon^n\right\|_0^2\right)+\frac{C\triangle t^2}{RePr}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2dt. \end{aligned} \tag{6.4.62}$$

By choosing $\psi_h=2\triangle t\chi_h^{n+1}$ in (6.4.33), we have

$$\begin{aligned} &2\left\langle\varepsilon^{n+1}-\varepsilon^n\,,\,\chi_h^{n+1}\right\rangle+\frac{2\triangle t}{RePr}\left\langle\nabla\varepsilon^{n+1}\,,\,\nabla\chi_h^{n+1}\right\rangle \\ &=2\triangle t\left\langle R_\theta^{n+1}\,,\,\chi_h^{n+1}\right\rangle-2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\chi_h^{n+1}\right) \\ &\quad+2\triangle t\mathcal{N}\left(\mathbf{u}_h^{n+1},\theta_h^{n+1},\chi_h^{n+1}\right). \end{aligned} \tag{6.4.63}$$

The left hand side terms can be written as follows, by (6.4.50),

$$\begin{aligned} 2\left\langle\varepsilon^{n+1}-\varepsilon^n\,,\,\chi_h^{n+1}\right\rangle &= 2\left\langle\nabla\left(\chi_h^{n+1}-\chi_h^n\right),\,\nabla\chi_h^{n+1}\right\rangle \\ &= \left\|\nabla\chi_h^{n+1}\right\|_0^2-\|\nabla\chi_h^n\|_0^2+\left\|\nabla\left(\chi_h^{n+1}-\chi_h^n\right)\right\|_0^2, \end{aligned} \tag{6.4.64}$$

and

$$\frac{2\triangle t}{RePr}\left\langle \nabla\varepsilon^{n+1}\,,\,\nabla\chi_h^{n+1}\right\rangle = \frac{2\triangle t}{RePr}\left(\left\langle \varepsilon^{n+1}\,,\,\varepsilon_h^{n+1}\right\rangle + \left\langle \nabla\delta^{n+1}\,,\,\nabla\chi_h^{n+1}\right\rangle\right)$$
$$= \frac{2\triangle t}{RePr}\left(\left\|\varepsilon^{n+1}\right\|_0^2 - \left\langle \varepsilon^{n+1}\,,\,\delta^{n+1}\right\rangle + \left\langle \nabla\delta^{n+1}\,,\,\nabla\chi_h^{n+1}\right\rangle\right). \tag{6.4.65}$$

So (6.4.63) becomes

$$\left\|\nabla\chi_h^{n+1}\right\|_0^2 - \left\|\nabla\chi_h^{n}\right\|_0^2 + \left\|\nabla\left(\chi_h^{n+1}-\chi_h^n\right)\right\|_0^2 + \frac{2\triangle t}{RePr}\left\|\varepsilon^{n+1}\right\|_0^2$$
$$= 2\triangle t\left\langle R_\theta^{n+1}\,,\,\chi_h^{n+1}\right\rangle + \frac{2\triangle t}{RePr}\left(\left\langle \varepsilon^{n+1}\,,\,\delta^{n+1}\right\rangle - \left\langle \nabla\delta^{n+1}\,,\,\nabla\chi_h^{n+1}\right\rangle\right)$$
$$-2\triangle t\left(\mathcal{N}\left(\mathbf{u}(t_{n+1}),\theta(t_{n+1}),\chi_h^{n+1}\right) - \mathcal{N}\left(\mathbf{u}_h^{n+1},\theta_h^{n+1},\chi_h^{n+1}\right)\right). \tag{6.4.66}$$
$$= A_7 + A_8 + A_9 + A_{10}.$$

We now estimate each split term as follow,

$$A_7 \le \frac{C\triangle t^2}{Re}\int_{t_n}^{t_{n+1}} \|\theta_{tt}(t)\|_{-1}^2 dt + CRe\triangle t\left\|\nabla\chi_h^{n+1}\right\|_0^2, \tag{6.4.67}$$

$$A_8 \le \frac{\triangle t}{4RePr}\left\|\varepsilon^{n+1}\right\|_0^2 + \frac{C\triangle t}{RePr}\left\|\delta^{n+1}\right\|_0^2, \tag{6.4.68}$$

and

$$A_9 \le \frac{C\triangle th}{RePr}\left\|\nabla\delta^{n+1}\right\|_0\left\|\chi^{n+1}\right\|_2 + \frac{C\triangle t}{RePr}\left\|\delta^{n+1}\right\|_0\left\|\chi^{n+1}\right\|_2$$
$$\le \frac{C\triangle th^2}{RePr}\left\|\nabla\delta^{n+1}\right\|_0^2 + \frac{C\triangle t}{RePr}\left\|\delta^{n+1}\right\|_0^2 + \frac{\triangle t}{4RePr}\left\|\varepsilon^{n+1}\right\|_0^2. \tag{6.4.69}$$

And the last term in (6.4.66) can be split again by

$$A_{10} = -2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1},\theta(t_{n+1}),\chi_h^{n+1}\right)$$
$$-2\triangle t\mathcal{N}\left(\mathbf{u}_h^{n+1},\varepsilon^{n+1},\chi_h^{n+1}\right) = A_{10,1} + A_{10,2}. \tag{6.4.70}$$

The first term in (6.4.70) can be estimated simply by

$$A_{10,1} \le C\triangle t\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\right\|_0\|\theta(t_{n+1})\|_2\left\|\nabla\chi_h^{n+1}\right\|_0$$
$$\le \frac{\triangle t}{2Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2\right) + CRe\triangle t\left\|\nabla\chi_h^{n+1}\right\|_0^2, \tag{6.4.71}$$

and the second term split again

$$
\begin{aligned}
A_{10,2} &= 2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}, \varepsilon^{n+1}, \chi_h^{n+1}\right) \\
&\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}), \varepsilon^{n+1}, \chi_h^{n+1}\right) \\
&= 2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}, \varepsilon^{n+1}, \chi_h^{n+1} - \chi^{n+1}\right) \\
&\quad +2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}, \varepsilon^{n+1}, \chi^{n+1}\right) \\
&\quad -2\triangle t\mathcal{N}\left(\mathbf{u}(t_{n+1}), \varepsilon^{n+1}, \chi_h^{n+1}\right) = A_{10,3} + A_{10,4} + A_{10,5}.
\end{aligned} \tag{6.4.72}
$$

Lemma 6.9 and formula (6.4.54) yield

$$
\begin{aligned}
A_{10,3} &\leq C\triangle t\left\|\mathbf{E}^{n+1} + \mathbf{G}^{n+1}\right\|_0 \left\|\nabla\varepsilon^{n+1}\right\|_0 \left\|\chi_h^{n+1} - \chi^{n+1}\right\| \\
&\leq C\triangle t\left(\triangle t^{\frac{1}{2}} + h^s\right)\left\|\nabla\varepsilon^{n+1}\right\|_0 \left\|\varepsilon^{n+1}\right\|_0 \\
&\leq CRePr\triangle t\left(\triangle t + h^{2s}\right)\left\|\nabla\varepsilon^{n+1}\right\|_0^2 + \frac{\triangle t}{6RePr}\left\|\varepsilon^{n+1}\right\|_0^2,
\end{aligned} \tag{6.4.73}
$$

$$
\begin{aligned}
A_{10,4} &\leq C\triangle t\left\|\mathbf{E}^{n+1} + \mathbf{G}^{n+1}\right\|_0 \left\|\nabla\varepsilon^{n+1}\right\|_0 \left\|\chi^{n+1}\right\|_2 \\
&\leq C\triangle t\left(\triangle t^{\frac{1}{2}} + h^s\right)\left\|\nabla\varepsilon^{n+1}\right\|_0 \left\|\varepsilon^{n+1}\right\|_0 \\
&\leq CRePr\triangle t\left(\triangle t + h^{2s}\right)\left\|\nabla\varepsilon^{n+1}\right\|_0^2 + \frac{\triangle t}{6RePr}\left\|\varepsilon^{n+1}\right\|_0^2,
\end{aligned} \tag{6.4.74}
$$

and

$$
\begin{aligned}
A_{10,5} &\leq C\triangle t\|\mathbf{u}(t_{n+1})\|_2 \left\|\varepsilon^{n+1}\right\|_0 \left\|\nabla\chi_h^{n+1}\right\|_0 \\
&\leq \frac{\triangle t}{6RePr}\left\|\varepsilon^{n+1}\right\|_0^2 + CRePr\triangle t\left\|\nabla\chi_h^{n+1}\right\|_0^2.
\end{aligned} \tag{6.4.75}
$$

Upon gathering (6.4.70)-(6.4.75), we derive

$$
\begin{aligned}
A_{10} \leq\ & \frac{\triangle t}{2Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2\right) + \frac{\triangle t}{2RePr}\left\|\varepsilon^{n+1}\right\|_0^2 \\
& +CRe\triangle t\left\|\nabla\chi_h^{n+1}\right\|_0^2 + CRePr\triangle t\left(\triangle t + h^{2s}\right)\left\|\nabla\varepsilon^{n+1}\right\|_0^2.
\end{aligned} \tag{6.4.76}
$$

Plugging (6.4.67)-(6.4.69) and (6.4.76) into (6.4.66) yields

$$\begin{aligned}
&\left\|\nabla \chi_h^{n+1}\right\|_0^2-\|\nabla \chi_h^n\|_0^2+\left\|\nabla\left(\chi_h^{n+1}-\chi_h^n\right)\right\|_0^2+\frac{\triangle t}{RePr}\left\|\varepsilon^{n+1}\right\|_0^2 \\
&\leq \frac{\triangle t}{2Re}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)+CRe\triangle t\left\|\nabla \chi_h^{n+1}\right\|_0^2 \\
&\quad+\frac{C\triangle t}{RePr}\left\|\delta^{n+1}\right\|_0^2+\frac{C\triangle t h^2}{RePr}\left\|\nabla \delta^{n+1}\right\|_0^2 \\
&\quad+CRePr\triangle t\left(\triangle t+h^{2s}\right)\left\|\nabla \varepsilon^{n+1}\right\|_0^2+\frac{C\triangle t^2}{Re}\int_{t_n}^{t_{n+1}}\|\theta_{tt}(t)\|_{-1}^2 dt.
\end{aligned} \tag{6.4.77}$$

By adding (6.4.62) and (6.4.77), we get

$$\begin{aligned}
&\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2-\|\nabla \mathbf{v}_h^n\|_0^2+\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+\frac{\triangle t}{2Re}\left\|\mathbf{E}^{n+1}\right\|_0^2 \\
&+\left\|\nabla \chi_h^{n+1}\right\|_0^2-\|\nabla \chi_h^n\|_0^2+\left\|\nabla\left(\chi_h^{n+1}-\chi_h^n\right)\right\|_0^2+\frac{\triangle t}{2RePr}\left\|\varepsilon^{n+1}\right\|_0^2 \\
&\leq CRe\triangle t\left(\left\|\nabla \mathbf{v}_h^{n+1}\right\|_0^2+\left\|\nabla \chi_h^{n+1}\right\|_0^2+h^2\left\|\eta^{n+1}\right\|_0^2\right) \\
&\quad+\frac{C\triangle t}{Re}\left(h^2\left\|\nabla \mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2\right. \\
&\quad\left.+\left\|\mathbf{G}^{n+1}\right\|_0^2+\|\mathbf{G}^n\|_0^2+\left\|\nabla \rho_h^{n+1}\right\|_0^2\right)+\frac{CGr^2Pr\triangle t}{Re^3}\left\|\mathbf{v}_h^{n+1}\right\|_0^2 \\
&\quad+CRe\triangle t(\triangle t+h^{2s})\left(\left\|\nabla \mathbf{G}^{n+1}\right\|_0^2+\left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\
&\quad+\frac{C\triangle t}{RePr}\left(\left\|\varepsilon^{n+1}-\varepsilon^n\right\|_0^2+\left\|\delta^{n+1}\right\|_0^2+h^2\left\|\nabla \delta^{n+1}\right\|_0^2\right) \\
&\quad+CRePr\triangle t\left(\triangle t+h^{2s}\right)\left\|\nabla \varepsilon^{n+1}\right\|_0^2+\frac{C\triangle t^2}{RePr}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2 dt \\
&\quad+\frac{C\triangle t^2}{Re}\int_{t_n}^{t_{n+1}}\|\theta_{tt}(t)\|_{-1}^2 dt+C\triangle t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt.
\end{aligned} \tag{6.4.78}$$

On adding over n from 0 to N yields

$$
\begin{aligned}
&\left\|\nabla\mathbf{v}_h^{N+1}\right\|_0^2+\left\|\nabla\chi_h^{N+1}\right\|_0^2+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2+\frac{\triangle t}{2RePr}\sum_{n=0}^{N}\left\|\varepsilon^{n+1}\right\|_0^2\\
&+\sum_{n=0}^{N}\left(\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\|_0^2+\left\|\nabla\left(\chi_h^{n+1}-\chi_h^{n}\right)\right\|_0^2\right)\\
&\leq CRe\triangle t\sum_{n=0}^{N}\left(\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2+\left\|\nabla\chi_h^{n+1}\right\|_0^2+h^2\left\|\eta^{n+1}\right\|_0^2\right)\\
&\quad+\frac{C\triangle t}{Re}\sum_{n=0}^{N}\left(h^2\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{F}^{n+1}\right\|_0^2+\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2\right.\\
&\quad\left.+\left\|\mathbf{G}^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n}\right\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2\right)+\frac{CGr^2Pr\triangle t}{Re^3}\sum_{n=0}^{N}\left\|\mathbf{v}_h^{n+1}\right\|_0^2\\
&\quad+CRe\triangle t(\triangle t+h^{2s})\sum_{n=0}^{N}\left(\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right)\\
&\quad+\frac{C\triangle t}{RePr}\sum_{n=0}^{N}\left(\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_0^2+\left\|\delta^{n+1}\right\|_0^2+h^2\left\|\nabla\delta^{n+1}\right\|_0^2\right)\\
&\quad+CRePr\triangle t\left(\triangle t+h^{2s}\right)\sum_{n=0}^{N}\left\|\nabla\varepsilon^{n+1}\right\|_0^2\\
&\quad+C\triangle t^2\int_{t_0}^{t_{N+1}}\left(\frac{1}{RePr}\|\theta_t(t)\|_0^2+\frac{1}{Re}\|\theta_{tt}(t)\|_{-1}^2+\|\mathbf{u}_t(t)\|_0^2\right)dt.
\end{aligned}
\tag{6.4.79}
$$

We note that Assumptions 7 and 8 imply $\mathbf{v}_h^0=0$ and $\chi_h^0=0$. By Lemmas 5.9, 6.5, 6.6, 6.9, 6.10, and discrete Gronwall lemma, we have

$$
\begin{aligned}
&\left\|\mathbf{E}^{N+1}\right\|_{\mathbf{Z}^*}^2+\left\|\varepsilon^{N+1}\right\|_{-1}^2+\frac{\triangle t}{2Re}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2+\frac{\triangle t}{2RePr}\sum_{n=0}^{N}\left\|\varepsilon^{n+1}\right\|_0^2\\
&\quad+\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_{\mathbf{Z}^*}^2+\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_{-1}^2\right)\leq C\left(\triangle t^2+h^{2s+2}\right).
\end{aligned}
\tag{6.4.80}
$$

Also, by Lemmas 6.7 and 6.9, we see that

$$\begin{aligned}\triangle t\sum_{n=0}^{N}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 &\le \triangle t\sum_{n=0}^{N}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2\right)\\ &\le C\left(\triangle t^2+h^{2s+2}\right).\end{aligned} \tag{6.4.81}$$

which concludes the proof Lemma 6.11. ■

6.5 Error Estimate for Pressure

The main goal in this section to show the improved estimate

$$\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2 \le C\left(\triangle t^2+h^{2s+2}\right). \tag{6.5.1}$$

We begin by estimating the error of the first iteration time step.

Lemma 6.12 *Let Assumptions 1-9 hold, and let $h^2 \le C\triangle t$. If*

$$\left\|\nabla\mathbf{u}_t(0)\right\|_0 \le M \quad \text{and} \quad \sup_{0\le t\le t_1}\left\|\theta_{tt}(t)\right\|_{-1} \le M \tag{6.5.2}$$

then we have

$$\begin{aligned}\left\|\mathbf{E}^1\right\|_0^2+\frac{1}{2}\left(\left\|\mathbf{E}^1-\mathbf{E}^0\right\|_0^2+\left\|\varepsilon^1-\varepsilon^0\right\|_0^2\right)+\frac{\triangle t}{2Re}\left(\left\|\nabla\widehat{\mathbf{E}}^1\right\|_0^2+\left\|s_h^1\right\|_0^2\right)\\ +\left\|\nabla\rho_h^1\right\|_0^2+\left\|\varepsilon^1\right\|_0^2+\frac{\triangle t}{RePr}\left\|\nabla\varepsilon^1\right\|_0^2 \le C\left(\triangle t^2+h^{2s+2}\right)\end{aligned} \tag{6.5.3}$$

and

$$\begin{aligned}\left\|\nabla\mathbf{v}_h^1\right\|_0^2+\frac{3\triangle t}{4Re}\left\|\mathbf{E}^1\right\|_0^2+\left\|\nabla\chi_h^1\right\|_0^2+\frac{\triangle t}{RePr}\left\|\varepsilon^1\right\|_0^2\\ \le C\triangle t\left(\triangle t^2+h^{2s+2}\right).\end{aligned} \tag{6.5.4}$$

PROOF. We note $s_h^0 = 0$ and $\mathbf{v}_h^0 = 0$ by Assumption 7 and $\chi_h^0 = 0$ by Assumption 8. By choosing $n = 0$ in (6.4.47), we have

$$
\begin{aligned}
&\|\mathbf{E}^1\|_0^2 + \frac{1}{2}\left(\|\mathbf{E}^1 - \mathbf{E}^0\|_0^2 + \|\varepsilon^1 - \varepsilon^0\|_0^2\right) + \frac{\triangle t}{2Re}\left(\left\|\nabla\widehat{\mathbf{E}}^1\right\|_0^2 + \|s_h^1\|_0^2\right) \\
&+\|\nabla\rho_h^1\|_0^2 + \|\varepsilon^1\|_0^2 + \frac{\triangle t}{RePr}\|\nabla\varepsilon^1\|_0^2 \leq \|\mathbf{E}^0\|_0^2 + \|\varepsilon^0\|_0^2 + \frac{C\triangle t}{RePr}\|\nabla\delta^1\|_0^2 \\
&\quad +CRePr\triangle t\left(\|\mathbf{E}^1\|_0^2 + \|\mathbf{G}^1\|_0^2\right) + C\|\mathbf{F}^1\|_0^2 + \frac{C\triangle t}{Re}\|\nabla\mathbf{F}^1\|_0^2 \\
&\quad +CRe\triangle t\left(\|\mathbf{E}^0\|_0^2 + \|\mathbf{G}^0\|_0^2 + \|\mathbf{G}_h^{n+1}\|_0^2 + \|\eta^{n+1}\|_0^2\right) \\
&\quad +C\triangle t^2\left(\|\mathbf{u}(t_1)\|_2^2 + \|\nabla p(t_1)\|_0^2\right) + CRePr\triangle t^2\int_{t_0}^{t_1}\|\theta_{tt}(t)\|_{-1}^2 dt \\
&\quad +C\|\delta^1\|_0^2 + \frac{C\triangle tGr}{Re^2}\left(\|\varepsilon^0\|_0^2 + \|\mathbf{F}^1\|_0^2 + \left\|\widehat{\mathbf{E}}^1\right\|_0^2\right) \\
&\quad +CRe\triangle t^2\int_{t_0}^{t_1}\|\mathbf{u}_t(t)\|_0^2 dt + \frac{C\triangle t^2 Gr}{Re^2}\int_{t_0}^{t_1}\|\theta_t(t)\|_0^2 dt.
\end{aligned}
\tag{6.5.5}
$$

So we get (6.5.3) by the results in Section 6.4. Because $\|\mathbf{E}^1\|_0^2 \leq C\triangle t + h^{2s}$ by Lemma 6.9, we do not need to choose small $\triangle t$ in initial step. And by choosing $n = 0$ in (6.4.78), we have

$$
\begin{aligned}
&\|\nabla\mathbf{v}_h^1\|_0^2 + \|\nabla\chi_h^1\|_0^2 \leq CRe\triangle t\left(\|\nabla\mathbf{v}_h^1\|_0^2 + \|\nabla\chi_h^1\|_0^2 + h^2\|\eta^1\|_0^2\right) \\
&\quad +\frac{C\triangle t}{Re}\left(h^2\|\nabla\mathbf{F}^1\|_0^2 + \|\mathbf{F}^1\|_0^2 + \|\mathbf{E}^1 - \mathbf{E}^0\|_0^2 + \|\mathbf{G}^1\|_0^2 + \|\mathbf{G}^0\|_0^2\right. \\
&\quad \left.+\|\nabla\rho_h^1\|_0^2\right) + CRe\triangle t(\triangle t + h^{2s})\left(\|\nabla\mathbf{G}^1\|_0^2 + \left\|\nabla\widehat{\mathbf{E}}^1\right\|_0^2\right) \\
&\quad +\frac{C\triangle t}{RePr}\left(\|\varepsilon^1 - \varepsilon^0\|_0^2 + \|\delta^1\|_0^2 + h^2\|\nabla\delta^1\|_0^2\right) \\
&\quad +\frac{CGr^2Pr\triangle t}{Re^3}\|\mathbf{v}_h^1\|_0^2 + CRePr\triangle t\left(\triangle t + h^{2s}\right)\|\nabla\varepsilon^1\|_0^2 \\
&\quad +C\triangle t^2\int_{t_0}^{t_1}\left(\frac{1}{RePr}\|\theta_t(t)\|_0^2 + \frac{1}{Re}\|\theta_{tt}(t)\|_{-1}^2 + \|\mathbf{u}_t(t)\|_0^2\right) dt.
\end{aligned}
\tag{6.5.6}
$$

Since $\|\nabla\mathbf{v}_h^1\|_0^2 \leq \|\mathbf{E}^1\|_0^2 \leq C(h^2 + h^{2s+2})$, we get (6.5.4) by the results in Section 6.4 and (6.5.3). ∎

Lemma 6.13 *Let Assumptions 1-9 hold, If* $C_1 h^2 \le \triangle t \le C_2 h^{\frac{2d}{3}}$ *with arbitrary* $C_1,\, C_2 > 0$, *and if*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le M, \tag{6.5.7}$$

then we have

$$\begin{aligned} &\left\|\mathbf{E}^{N+1} - \mathbf{E}^N\right\|_0^2 + \frac{\triangle t}{Re}\left\|s_h^{N+1} - s_h^N\right\|_0^2 + \sum_{n=1}^{N}\left\|\nabla\left(\rho_h^{n+1} - \rho_h^n\right)\right\|_0^2 \\ &+ \frac{\triangle t}{2Re}\sum_{n=1}^{N}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right)\right\|_0^2 + \frac{1}{2}\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right\|_0^2 \\ &\le C\left(\triangle t^2 + h^{2s+2}\right). \end{aligned} \tag{6.5.8}$$

Proof. Upon subtracting two consecutive expressions of (6.4.26), we have

$$\begin{aligned} &\left\langle \frac{\widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n}{\triangle t} - \frac{\widehat{\mathbf{E}}^n - \mathbf{E}^{n-1}}{\triangle t}\,,\, \mathbf{w}_h \right\rangle + \frac{1}{Re}\left\langle \nabla\left(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right),\, \nabla \mathbf{w}_h \right\rangle \\ &= \left\langle P^{n+1} - P^n\,,\, \mathrm{div}\,\mathbf{w}_h \right\rangle - \frac{1}{Re}\left\langle s_h^n - s_h^{n-1}\,,\, \mathrm{div}\,\mathbf{w}_h \right\rangle \\ &\quad -\mathcal{N}(u(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{w}_h) + \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h \\ &\quad +\mathcal{N}(\mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{w}_h) - \mathcal{N}(\mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^n, \mathbf{w}_h) \\ &\quad -\frac{Gr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1}) - \theta_h^n) - \mathbf{g}(\theta(t_n) - \theta_h^{n-1})\,,\, \mathbf{w}_h \right\rangle \end{aligned} \tag{6.5.9}$$

We choose $\mathbf{w}_h = 2\triangle t\left(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right)$ in (6.5.9). By (5.4.43) and (5.4.44), the left hand side of (6.5.9) becomes

$$
\begin{aligned}
&\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 - \left\|\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0^2 + \left\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right\|_0^2 \\
&+2\left\|\nabla(\rho_h^{n+1} - \rho_h^n)\right\|_0^2 + \frac{2\triangle t}{Re}\left\|\nabla\left(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n\right)\right\|_0^2 \\
&= \frac{2\triangle t}{Re}\left\langle \nabla(\widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}^n)\,,\, \nabla(\mathbf{F}^{n+1} - \mathbf{F}^n)\right\rangle \\
&\quad +2\left\langle \mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\,,\, \mathbf{F}^{n+1} - \mathbf{F}^n\right\rangle \\
&\quad +2\triangle t\left\langle P^{n+1} - P^n\,,\, \mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\rangle \\
&\quad -\frac{2\triangle t}{Re}\left\langle s_h^n - s_h^{n-1}\,,\, \mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\rangle \\
&\quad -2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) - \mathcal{N}(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right) \\
&\quad +2\triangle t\left(\mathcal{N}(\mathbf{u}(t_n), \mathbf{u}(t_n), \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n) - \mathcal{N}(\mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^n, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right) \\
&\quad -\frac{2\triangle t Gr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1}) - \theta_h^n) - \mathbf{g}(\theta(t_n) - \theta_h^{n-1})\,,\, \widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n\right\rangle \\
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.
\end{aligned}
\tag{6.5.10}
$$

Then A_1-A_6 can be bounded by (5.4.46)- (5.4.63), and the last term is bounded by

$$
\begin{aligned}
A_7 \quad \leq \quad & \frac{C\triangle t^3 Gr^2}{Re^3}\int_{t_{n-1}}^{t_{n+1}} \|\theta_{tt}(t)\|_{-1}^2 dt + \frac{C\triangle t Gr^2}{Re^3}\left\|\varepsilon^n - \varepsilon^{n-1}\right\|_{-1}^2 \\
& + \frac{\triangle t}{8Re}\left\|\nabla(\widehat{\mathbf{E}}_h^{n+1} - \widehat{\mathbf{E}}_h^n)\right\|_0^2.
\end{aligned}
\tag{6.5.11}
$$

Since we have, by Lemma 6.11,

$$
\sum_{n=0}^{N}\left\|\varepsilon^{n+1} - \varepsilon^n\right\|_{-1}^2 \leq C\left(\triangle t^2 + h^{2s+2}\right), \tag{6.5.12}
$$

plugging (6.5.11) into (5.4.65) and treating by (5.4.66)-(5.4.72) derive (6.5.8). ■

Lemma 6.14 *Let Assumptions 1-9 hold. If $C_1 h^2 \le \triangle t \le C_2 h^{\frac{2d}{3}}$ with constants C_1, $C_2 > 0$, and if*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le M, \tag{6.5.13}$$

then we have

$$\begin{aligned}
&\left\|\nabla\left(\mathbf{v}_h^{N+1}-\mathbf{v}_h^{N}\right)\right\|_0^2+\frac{1}{2}\sum_{n=1}^{N}\left\|\nabla\left(\mathbf{v}_h^{n+1}-2\mathbf{v}_h^{n}+\mathbf{v}_h^{n-1}\right)\right\|_0^2\\
&+\frac{\triangle t}{Re}\sum_{n=1}^{N}\left\|\mathbf{E}_h^{n+1}-\mathbf{E}_h^{n}\right\|_0^2\le C\triangle t\left(\triangle t^2+h^{2s+2}\right).
\end{aligned}\tag{6.5.14}$$

Proof. We choose $\mathbf{w}_h = 2\triangle t\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)$ in (6.5.9), where $\mathbf{v}_h^{n+1}$ is the solution of the discrete Stokes system (5.3.46). Then we have

$$\begin{aligned}
&2\left\langle\left(\mathbf{E}^{n+1}-\mathbf{E}^{n}\right)-\left(\mathbf{E}^{n}-\mathbf{E}^{n-1}\right),\,\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right\rangle\\
&+\frac{2\triangle t}{Re}\left\langle\nabla\left(\widehat{\mathbf{E}}^{n+1}-\widehat{\mathbf{E}}^{n}\right),\,\nabla\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\rangle\\
&=2\triangle t\left\langle P^{n+1}-P^{n},\,\mathrm{div}\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\rangle\\
&\quad-2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})-\mathcal{N}(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right)\\
&\quad+2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n}),\mathbf{u}(t_{n}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})-\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right)\\
&\quad-\frac{2\triangle t Gr}{Re^2}\left\langle\mathbf{g}(\theta(t_{n+1})-\theta_h^{n})-\mathbf{g}(\theta(t_n)-\theta_h^{n-1}),\,\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right\rangle.
\end{aligned}\tag{6.5.15}$$

By (5.4.76) and (5.4.77), (6.5.15) becomes

$$
\begin{aligned}
&\left\|\nabla\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\|_0^2-\left\|\nabla\left(\mathbf{v}_h^{n}-\mathbf{v}_h^{n-1}\right)\right\|_0^2+\left\|\nabla\left(\mathbf{v}_h^{n+1}-2\mathbf{v}_h^{n}+\mathbf{v}_h^{n-1}\right)\right\|_0^2.\\
&+\frac{2\triangle t}{Re}\left\|\mathbf{E}^{n+1}-\mathbf{E}^{n}\right\|_0^2\\
&=\frac{2\triangle t}{Re}\left\langle\mathbf{E}^{n+1}-\mathbf{E}^{n}\,,\,\mathbf{F}^{n+1}-\mathbf{F}^{n}\right\rangle-\frac{2\triangle t}{Re}\left\langle q_h^{n+1}-q_h^{n}\,,\,\mathrm{div}\,(\widehat{\mathbf{E}}_h^{n+1}-\widehat{\mathbf{E}}_h^{n})\right\rangle\\
&\quad-\frac{2\triangle t}{Re}\left\langle\nabla(\mathbf{F}^{n+1}-\mathbf{F}^{n})\,,\,\nabla\left(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right)\right\rangle\\
&\quad+2\triangle t\left\langle P^{n+1}-P^{n}\,,\,\mathrm{div}\,(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\rangle\\
&\quad-2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})-\mathcal{N}(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right)\\
&\quad+2\triangle t\left(\mathcal{N}(\mathbf{u}(t_{n}),\mathbf{u}(t_{n}),\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})-\mathcal{N}(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right)\\
&\quad-\frac{2\triangle tGr}{Re^2}\left\langle\mathbf{g}(\theta(t_{n+1})-\theta_h^{n})-\mathbf{g}(\theta(t_{n})-\theta_h^{n-1})\,,\,\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right\rangle\\
&=A_1+A_2+A_3+A_4+A_5+A_6+A_7.
\end{aligned}
\tag{6.5.16}
$$

We already have A_1-A_6 in (5.4.79)-(5.4.101), so we need to get just

$$
\begin{aligned}
A_7 &= -\frac{2\triangle tGr}{Re^2}\left\langle\mathbf{g}\left((\theta(t_{n+1})-\theta_h^{n})-(\theta(t_n)-\theta_h^{n-1})\right)\,,\,\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n}\right\rangle\\
&\le \frac{C\triangle tGr}{Re^2}\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_{-1}^2+\frac{C\triangle tGr}{Re^2}\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^{n})\right\|_0^2\\
&\quad+\frac{C\triangle t^3Gr}{Re^2}\int_{t_{n-1}}^{t_{n+1}}\|\theta(t_{n+1})\|_{-1}^2dt.
\end{aligned}
\tag{6.5.17}
$$

Since, by Lemma 6.11,

$$
\sum_{n=0}^{N}\left\|\varepsilon^{n+1}-\varepsilon^{n}\right\|_{-1}^2\le C\left(\triangle t^2+h^{2s+2}\right),
\tag{6.5.18}
$$

plugging (6.5.18) into (5.4.102) gives (6.5.14). ■

Now we find the error of pressure by using all previous results:

Lemma 6.15 *Let Assumptions 1-9 hold. If* $C_1 h^2 \le \triangle t \le C_2 h^{\frac{2d}{3}}$ *with arbitrary constants* C_1, $C_2 > 0$, *and if*

$$\|\nabla \mathbf{u}_t(0)\|_0 \le M, \tag{6.5.19}$$

then we have

$$\triangle t \sum_{n=0}^{N} \left\|e_h^{n+1}\right\|_0^2 \le C\left(\triangle t + h^{2s}\right). \tag{6.5.20}$$

PROOF. In view of (6.1.8) in Algorithm 6.1, (6.4.26) can be rewritten by

$$\begin{aligned}
&\left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}\,,\,\mathbf{w}_h \right\rangle - \left\langle e^{n+1}\,,\,\mathrm{div}\,\mathbf{w}_h\right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla \mathbf{w}_h\right\rangle \\
&= \frac{1}{Re}\left\langle s_h^{n+1}-s_h^n\,,\,\mathrm{div}\,\mathbf{w}_h\right\rangle - \frac{Gr}{Re^2}\left\langle \mathbf{g}(\theta(t_{n+1})-\theta_h^n)\,,\,\mathbf{w}_h\right\rangle \\
&\quad -\mathcal{N}(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{w}_h) + \mathcal{N}(\mathbf{u}_h^n,\widehat{\mathbf{u}}_h^{n+1},\mathbf{w}_h).
\end{aligned} \tag{6.5.21}$$

By inf-sup Assumption 4, there exists an element $\mathbf{z}_h^{n+1} \in \mathbb{V}_h$ such that

$$\left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e_h^{n+1}\right\rangle = \left\|e_h^{n+1}\right\|_0^2 \quad \text{and} \quad \left\|\mathbf{z}_h^{n+1}\right\|_1 \le \frac{1}{\beta}\left\|e_h^{n+1}\right\|_0. \tag{6.5.22}$$

the above two formulas (6.5.21) and (6.5.22) yield

$$\begin{aligned}
\left\|e_h^{n+1}\right\|_0^2 &= \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e_h^{n+1}\right\rangle \\
&= \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,e^{n+1}+\eta^{n+1}\right\rangle \\
&= \left\langle \frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}\,,\,\mathbf{z}_h^{n+1}\right\rangle + \frac{1}{Re}\left\langle \nabla\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{z}_h^{n+1}\right\rangle \\
&\quad -\frac{1}{Re}\left\langle s_h^{n+1}-s_h^n\,,\,\mathrm{div}\,\mathbf{z}_h^{n+1}\right\rangle + \left\langle \mathrm{div}\,\mathbf{z}_h^{n+1}\,,\,\eta^{n+1}\right\rangle \\
&\quad +\left(\mathcal{N}\left(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{z}_h^{n+1}\right) - \mathcal{N}\left(\mathbf{u}_h^n,\widehat{\mathbf{u}}_h^{n+1},\mathbf{z}_h^{n+1}\right)\right) \\
&\quad +\frac{Gr}{Re^2}\left\langle \mathbf{g}\left(\theta(t_{n+1})-\theta_h^n\right)\,,\,\mathbf{z}_h^{n+1}\right\rangle \\
&= A_1+A_2+A_3+A_4+A_5+A_6.
\end{aligned} \tag{6.5.23}$$

Since we have already estimate A_1-A_5 in (5.4.109)-(5.4.118), it is enough to get

$$\begin{aligned} A_6 &= \frac{Gr}{Re^2}\left\langle \mathbf{g}\left(\theta(t_{n+1})-\theta_h^n\right), \mathbf{z}_h^{n+1}\right\rangle \\ &\leq \frac{CGr\triangle t}{\beta Re^4}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2 dt + \frac{CGr}{\beta Re^4}\|\varepsilon^n\|_0^2 + \frac{1}{4}\|\mathbf{e}_h^{n+1}\|_0^2. \end{aligned} \tag{6.5.24}$$

By (5.4.109)-(5.4.118) and (6.5.24), we obtain

$$\begin{aligned} \frac{1}{4}\|e_h^{n+1}\|_0^2 \leq\ & \frac{C}{\beta^2\triangle t^2}\|\mathbf{E}^{n+1}-\mathbf{E}^n\|_0^2 + \frac{C}{\beta^2 Re^2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &+\frac{Gr}{\beta Re^4}\|\varepsilon^n\|_0^2 + \frac{C}{\beta^2}\|\eta^{n+1}\|_0^2 \\ &+\frac{C}{\beta^2}\left(\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^{n+1}\|_0^2 + \|\mathbf{G}^n\|_0^2\right) \\ &+C\triangle t\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(t)\|_0^2 dt + \frac{CGr\triangle t}{\beta Re^4}\int_{t_n}^{t_{n+1}}\|\theta_t(t)\|_0^2 dt. \end{aligned} \tag{6.5.25}$$

Multiplying $4\triangle t$ and summation over n from 0 to N,

$$\begin{aligned} \triangle t\sum_{n=0}^{N}\|e_h^{n+1}\|_0^2 \leq\ & \frac{C}{\beta^2\triangle t}\sum_{n=0}^{N}\|\mathbf{E}^{n+1}-\mathbf{E}^n\|_0^2 + \frac{CGr}{\beta Re^4}\sum_{n=0}^{N}\|\varepsilon^n\|_0^2 \\ &+\frac{C\triangle t}{\beta^2}\sum_{n=0}^{N+1}\left(\left\|\widehat{\mathbf{E}}^n\right\|_0^2 + \|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \|\eta^n\|_0^2\right) \\ &+\frac{C\triangle t^2}{\beta^2}\int_{t_0}^{t_{N+1}}\left(\|\mathbf{u}_t(t)\|_0^2 + \frac{Gr}{Re^4}\|\theta_t(t)\|_0^2\right)dt \\ &+\frac{C\triangle t}{\beta^2 Re^2}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 \leq C\left(\triangle t + h^{2s}\right). \end{aligned} \tag{6.5.26}$$

By Lemmas 6.5, 6.9, and 6.14, we get (6.15). ■

6.6 Numerical Experiments

We present several simulations to illustrate the performance of the Gauge-Uzawa method with thermal convection. In all examples the fluid moves to buoyancy forces.

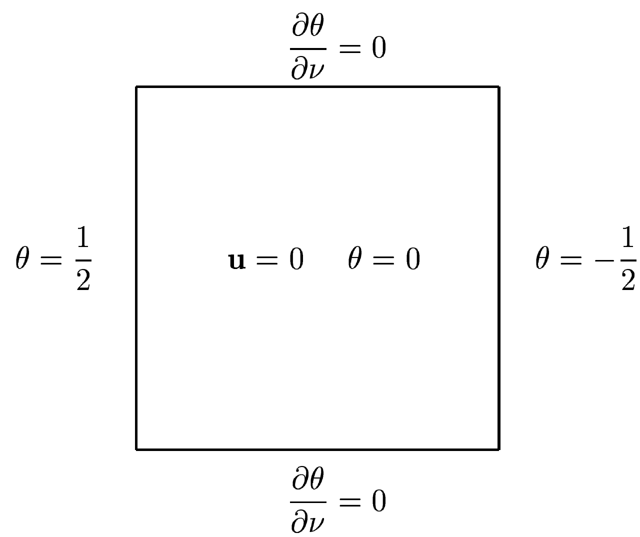


Figure 6.1: Initial and Boundary Values of Thermal Driven Cavity

Experiment 1 (Thermal Driven Cavity) Figure 6.1 indicates initial and boundary values in the domain $\Omega = [0,1]^2$. The non-dimensional numbers are chosen to be

$$Ra = 10^5, \quad Pr = 1, \quad Re = 1, \quad \triangle t = 0.0001, \quad h = \frac{1}{2^5}. \tag{6.6.1}$$

This example was already computed by Gresho, Lee, and Chan [13], and our results of Figures 6.2-6.6 are quite similar with theirs.

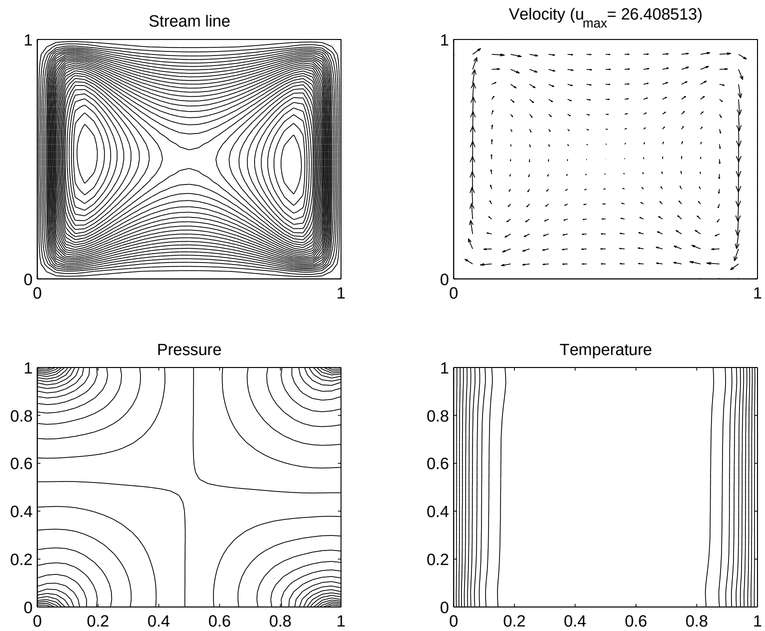


Figure 6.2: Thermal Driven Cavity of Experiment 1 at time=0.003.

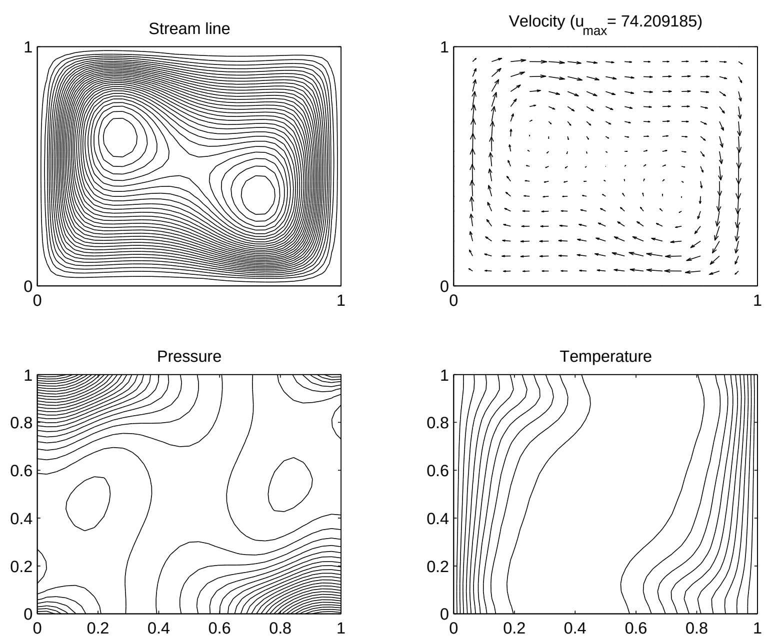


Figure 6.3: Thermal Driven Cavity of Experiment 1 at time=0.01.

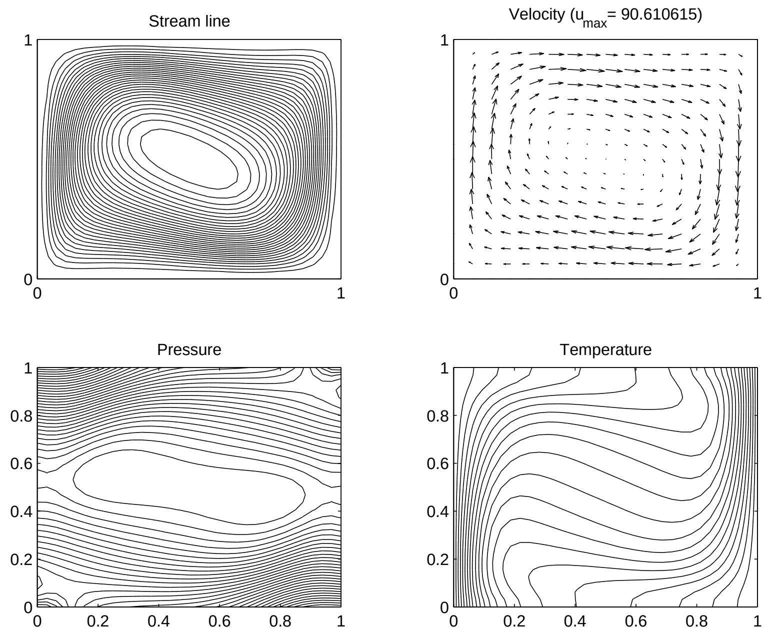


Figure 6.4: Thermal Driven Cavity of Experiment 1 at time=0.025.

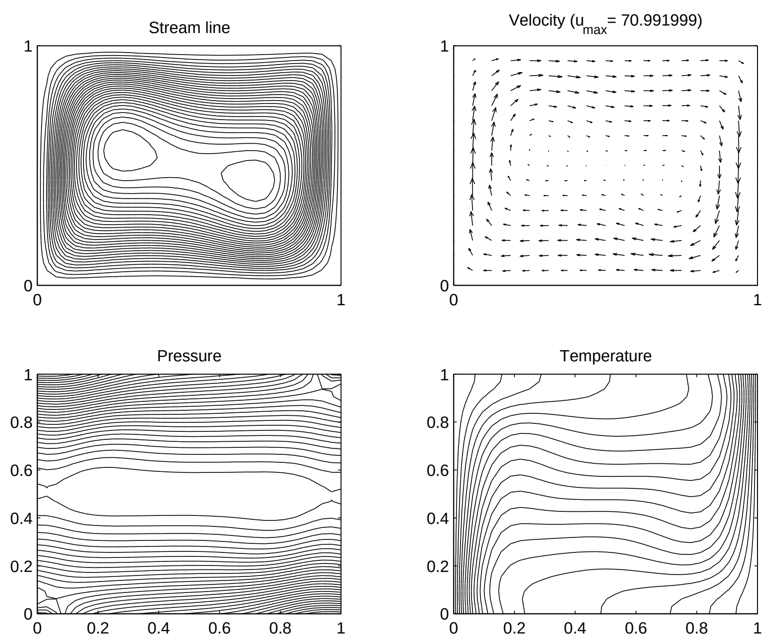


Figure 6.5: Thermal Driven Cavity of Experiment 1 at time=0.1.

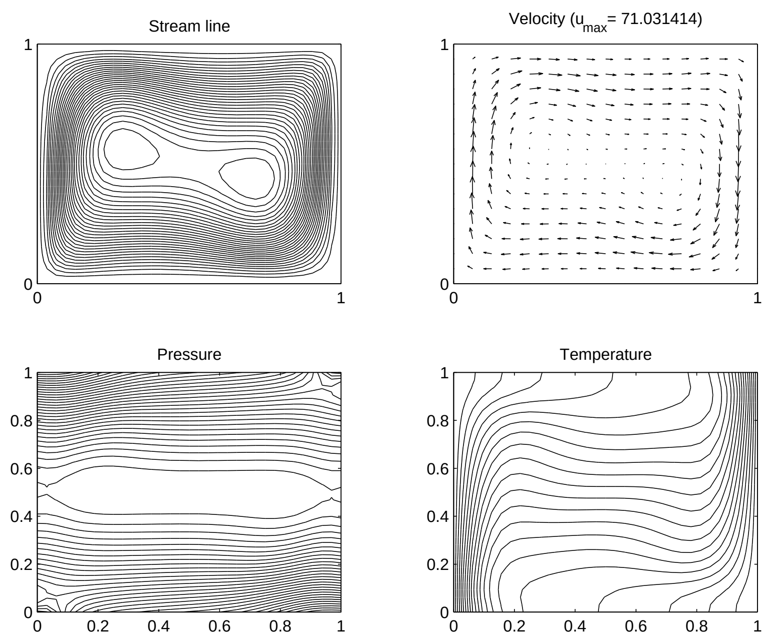


Figure 6.6: Thermal Driven Cavity of Experiment 1 at time=0.2.

Experiment 2 (Thermal Driven Cavity) This example is the same as Experiment 1 but with

$$Re = 10^4, \quad \triangle t = 1. \tag{6.6.2}$$

These extreme values are chosen to test the stability of the Gauge-Uzawa scheme. The remaining parameters are

$$Ra = 10^5, \quad Pr = 1, \quad h = 2^{-6}. \tag{6.6.3}$$

The results are depicted in Figure 6.7 to 6.11.

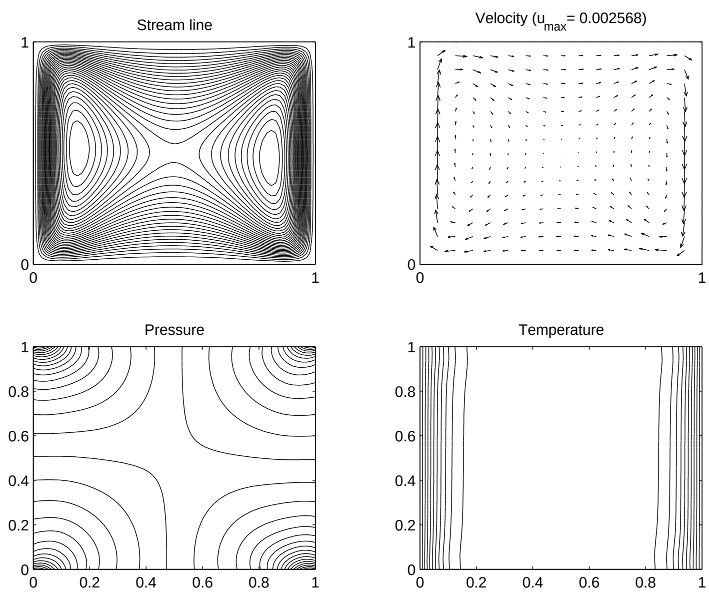


Figure 6.7: Thermal Driven Cavity of Experiment 2 at time=30.

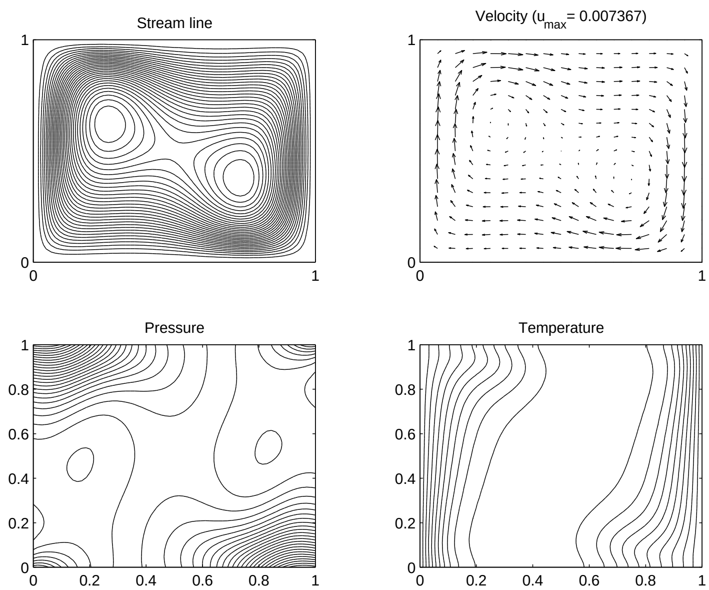


Figure 6.8: Thermal Driven Cavity of Experiment 2 at time=100.

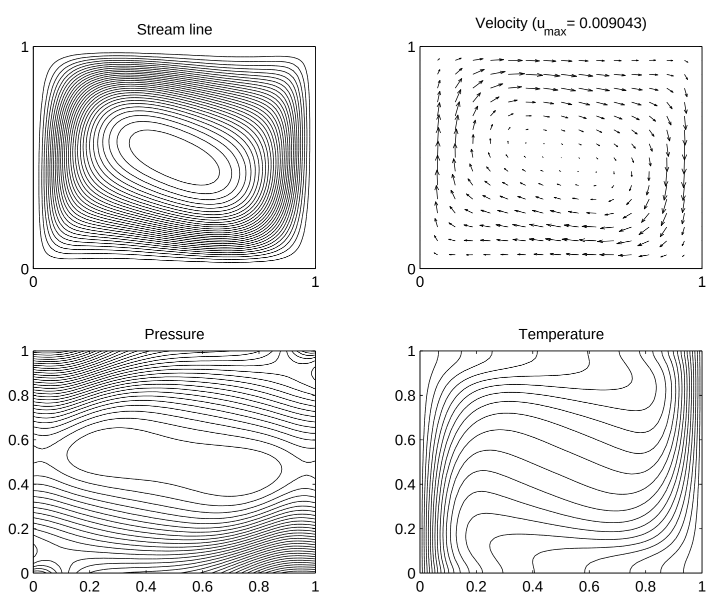


Figure 6.9: Thermal Driven Cavity of Experiment 2 at time=250.

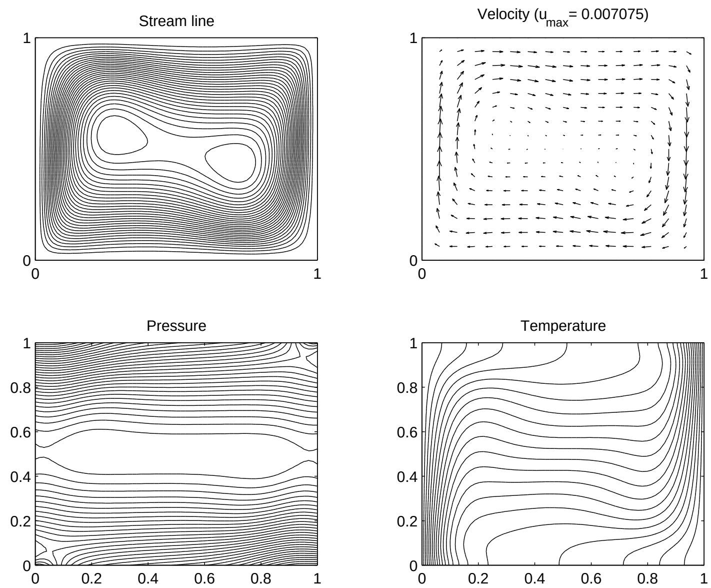


Figure 6.10: Thermal Driven Cavity of Experiment 2 at time=1000.

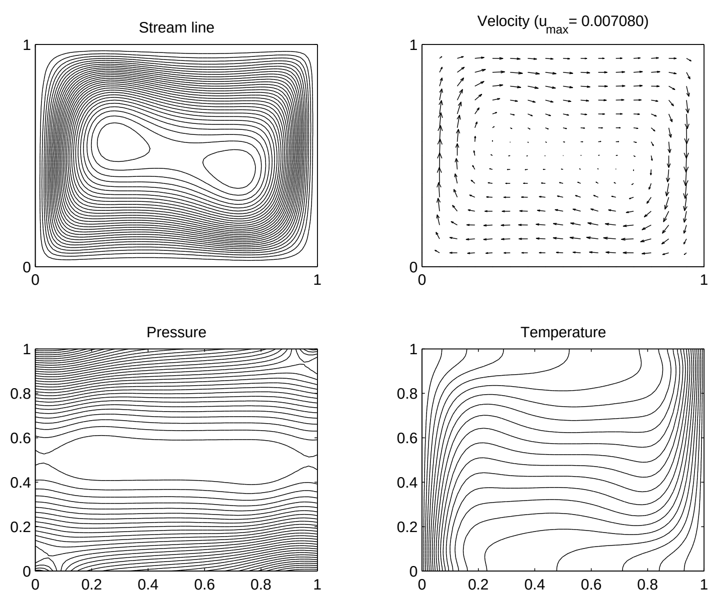


Figure 6.11: Thermal Driven Cavity of Experiment 2 at time=2000.

Experiment 3 (Benard Convection) The initial and boundary conditions are shown in Figure 6.12 for the domain $\Omega = [0,5] \times [0,1]$. The non-dimensional numbers are chosen to be

$$Ra = 10^4, \quad Pr = 1, \quad Re = 1, \tag{6.6.4}$$

with discretization parameters

$$\triangle t = 0.01, \quad h = \frac{1}{2^4}. \tag{6.6.5}$$

Figure 6.16 depict the solution towards steady state.

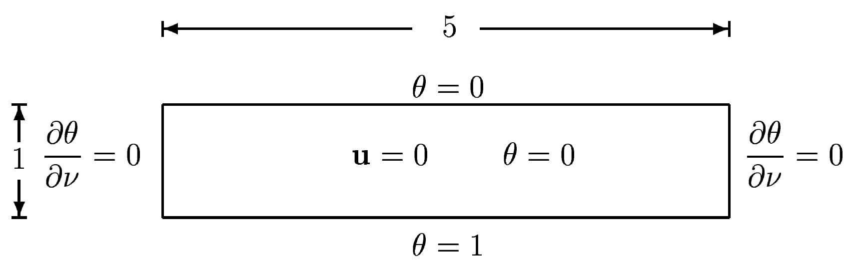


Figure 6.12: Initial and Boundary Values of Benard Example

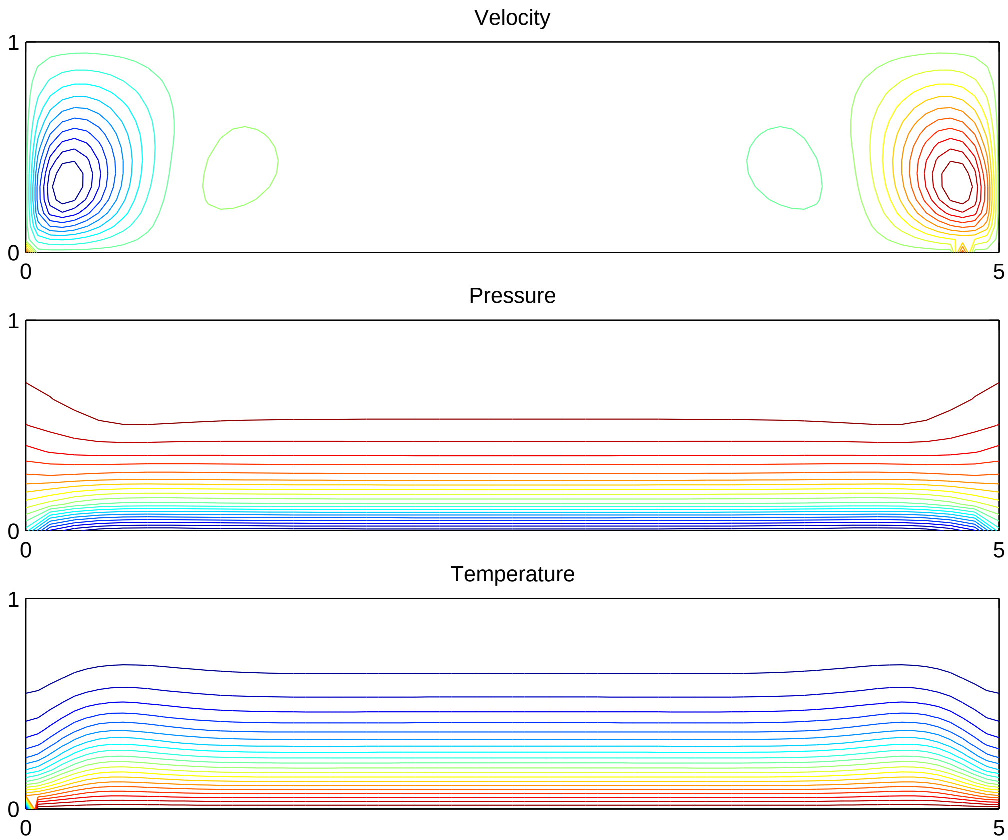


Figure 6.13: The Benard Example at $t = 0.05$

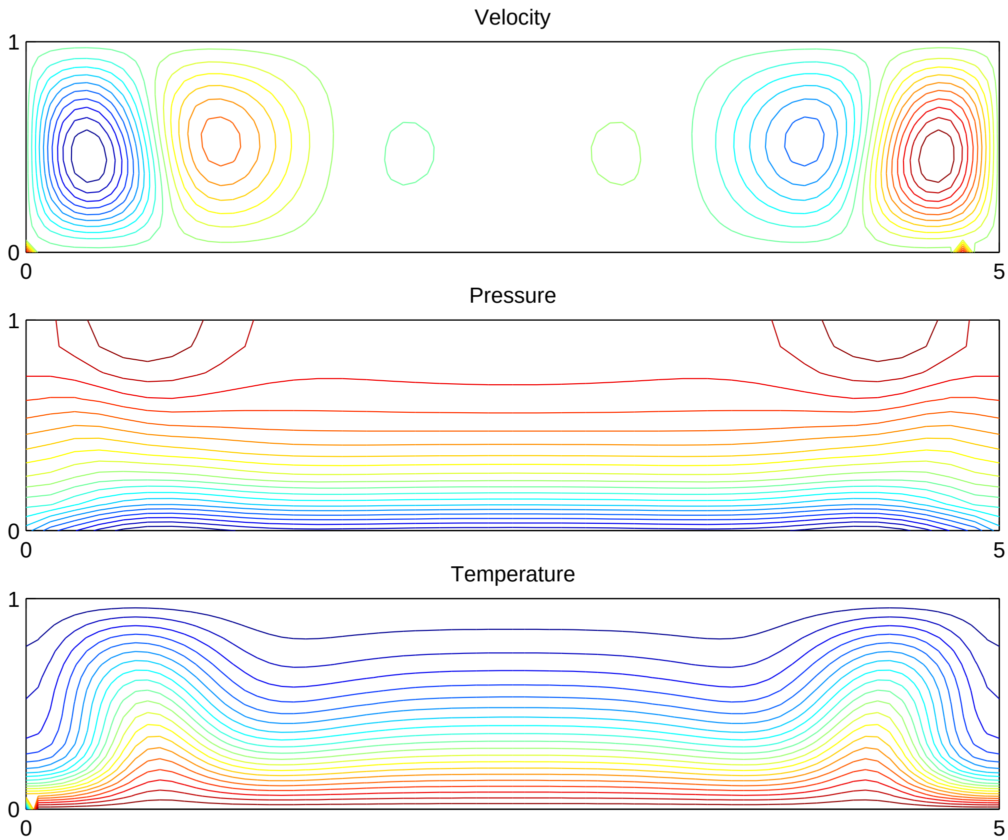


Figure 6.14: The Benard Example at $t = 0.10$

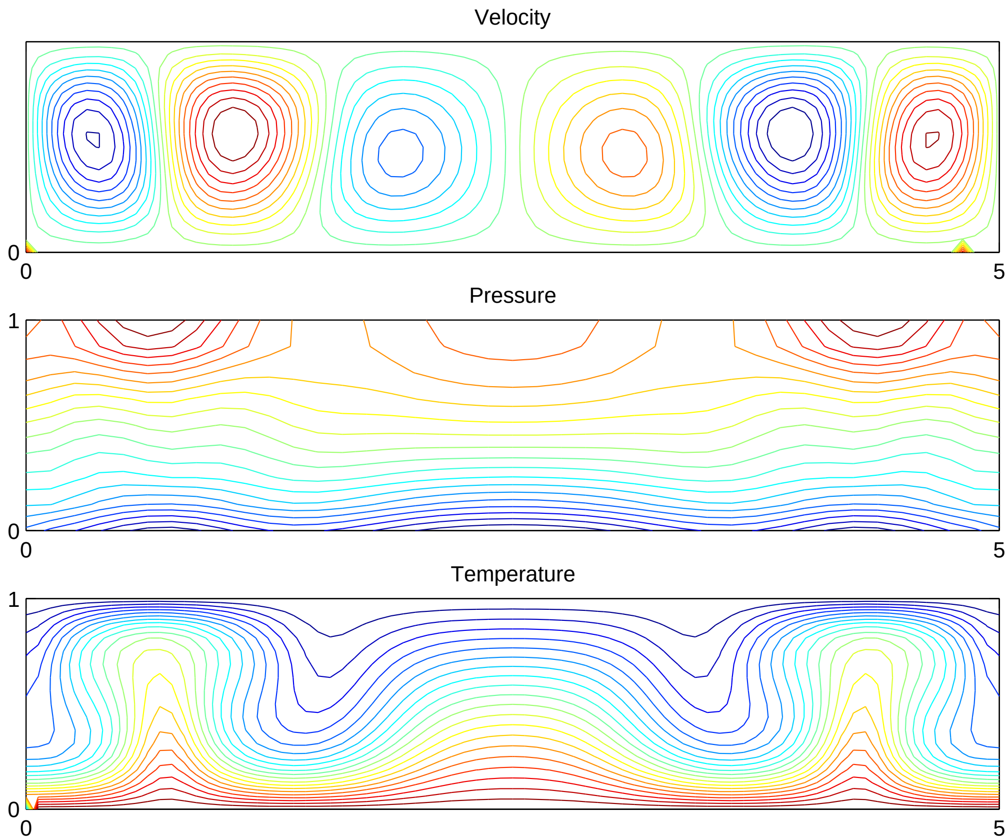


Figure 6.15: The Benard Example at $t = 0.15$

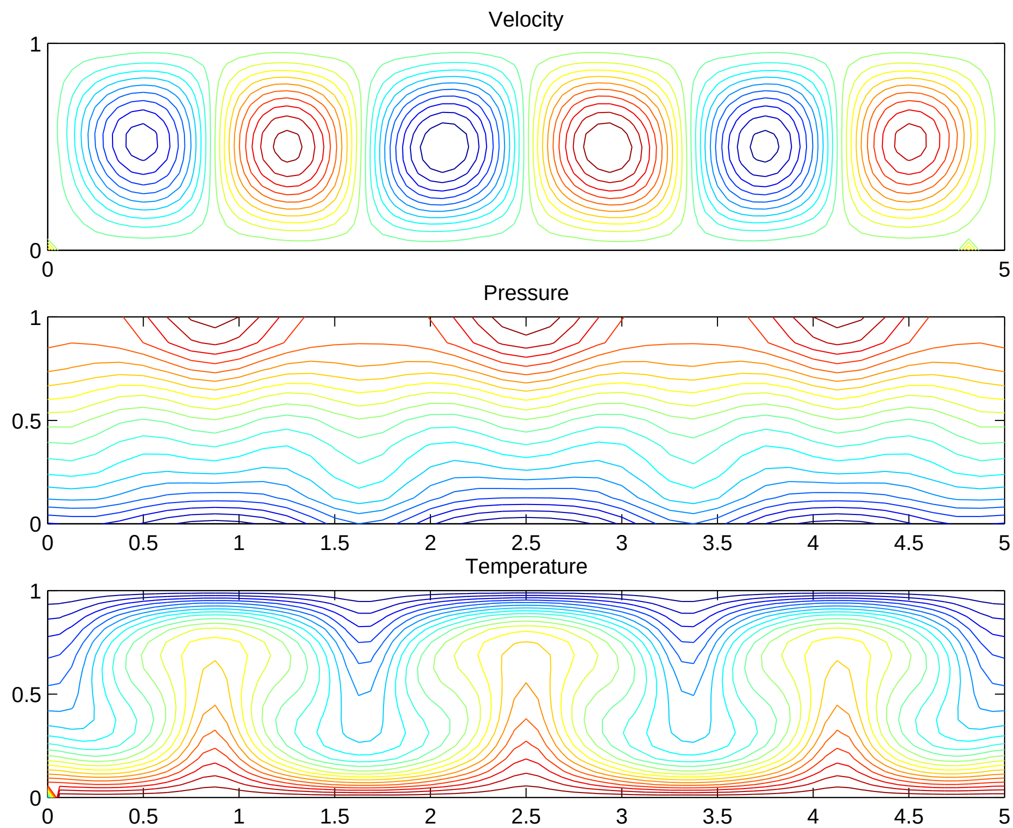


Figure 6.16: Steady State Solution of Benard Example $t = 1.0$

BIBLIOGRAPHY

[1] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, (1991).

[2] J.B. Bell, P. Colella, and H.M. Glaz, *A second-order projection method for the incompressible Navier-Stokes equations*, J. Comp. Phys., **85** (1989), 257-283.

[3] G.F. Carey, S.S. Chow and M.K. Seager, *Approximate boundary-flux calculations*, Comput. Meth. Appl. Mech. Eng., **50** (1985), 107-120.

[4] A.J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp., **22**, (1968) 745-762.

[5] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, (1988).

[6] W. E and J.-G. Liu, *Gauge finite element method for incompressible flows*, Int. J. Num. Meth. Fluids, **34** (2000), 701-710.

[7] W. E and J.-G. Liu, *Gauge method for viscous incompressible flows*, submitted to J. Comp. Phys., (1996).

[8] W. E and J.-G. Liu, *Projection method I: Convergence and numerical boundary layers*, SIAM J. Numer. Anal., **32** (1995), 1017–1057.

[9] H.C. Elman and G.H. Golub, *Inexact and preconditioned Uzawa algorithms for saddle-point problems*, SIAM J. Anal., **31**, No. 6 (1994), 1645-1661.

[10] L.C. Evans, *Partial Differential Equations*, GSM 19, American Mathematical Society, (1999).

[11] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order, Second Edition*, SCSM 224, Springer-Verlag, (1998).

[12] V. Girault, and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag (1986).

[13] P.M. Gresho, R.L. Lee, and S.T. Chan, *Solution of the time-dependent incompressible Navier-Stokes and Boussinesq equations Using the Galerkin finite element method*, Springer, Berlin, Lecture Notes in Math., **771** (1980), 203–222.

[14] J.-L. Guermond and L. Quartapelle, *On the approximation of the unsteady Navier-Stokes equations by finite element projection methods*, Numer. Math., **80** (1998), 207-238.

[15] J.-L. Guermond and L. Quartapelle, *On stability and convergence of projection methods based on pressure poison equation*, Int. J. Numer. Mech. Fluids, **26** (1998), 1039-1053.

[16] J.G. Heywood and R. Rannacher, *Finite element approximation of the non-stationary Navier-stokes problem. I. regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., **19**, no.2 (1982), 57-77.

[17] R.B. Kellogg and J.E. Osborn, *A regularity Result for the stokes problems in a convex polygon*, J. Funct. Anal., **21** (1976), 397-431.

[18] J. Kim and Mion, *Application of a fractional-step method to incompressible Navier-Stokes equations*, J.Comput. Phys, **59** (1985), 308-323.

[19] O. A. Ladyzenskaya, *In integral estimates, convergence, approximate methods, and solution in functionals for elliptic operators* Vestnik Leningrad University., **13**, no. 7 (1958), 60-69.

[20] A.I. Pehlivanov, R.D. Lazarov, G.F. Carey, and S.S. Chow, *Superconvergence analysis of approximate boundary-flux calculations*, Numer. Math., **63** (1992) 483-501.

[21] A. Prohl, *Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier-Stokes Equations*, B.G.Teubner Stuttgart (1997).

[22] A. Prohl, *On pressure approximation via projection methods in computational fluid dynamics*, pre-print.

[23] A. Schmidt and K.G. Siebert, *ALBERT: An adaptive hierarchical finite element toolbox* , Manual, 244 p., Preprint 06/2000 Freiburg.

[24] SFB 256, *GRAPE GRAphics Programming Environment Manual*, Universität Bonn (1995).

[25] J. Shen, *On error estimates of projection methods for Navier-Stokes equation: first order schemes*, SIAM J. Numer. Anal., **29** (1992), 57-77.

[26] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, Siam CBMS 66, (1995).

[27] R. Temam, *Sur l'approximation de la solution des equations de Navier-Stokes par la methode des pas fractionnaires. II. (French)* Arch. Rational Mech. Anal., **33** (1969), 377–385.

[28] Vidar Thomèe, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Springer Series in Comp. Math., (1997).

[29] R. Verfurth, *A posteriori error estimators or the Stokes equations*, Numer. Math., **55** (1989), 309-325.

[30] C. Wang and J.-G. Liu, *Convergence of gauge method for incompressible flow*, Math. Comp., **232** (2000), 1385–1407.

[31] M.F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations* SIAM J. Numer. Anal., **10** (1973), 723-759.