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NOTES ON NEW SINGULAR FUNCTION METHOD FOR DOMAIN SINGULARITIES

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Abstract. Recently, a new singular function(NSF) method was posed to get accurate numerical solution on quasi-uniform grids for two-dimensional Poisson and interface problems with domain singularities by the first author and his coworkers. Using the singular function representation of the solution, dual singular functions, and an extraction formula for stress intensity factors, the method poses a weak problem whose solution is in $H^2(\Omega)$ or $H^2(\Omega_i)$. In this paper, we show that the singular functions, which are not in $H^2(\Omega)$, also satisfy the integration by parts and note that this fact suggests the possibility of different choice of the weak formulations. We show that the original choice of weak formulation of NSF method is critical.

1. Introduction

We consider two types of partial differential equations defined on a non-convex polygonal domain $\Omega \subset \mathcal{R}^2$, whose solutions have the singular properties; Poisson problem with the Mixed boundary condition and the interface problem. Let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that Γ_D is not empty, i.e. $\text{meas}(\Gamma_D) \neq 0$. Let ν denote the outward unit vector normal to the boundary. For a given function $f \in L^2(\Omega)$, consider

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the Poisson equation with homogeneous mixed boundary conditions:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where Δ stands for the Laplace operator. We also assume that there is only one singular corner with type (D/N); boundary condition changes from the homogeneous Dirichlet boundary to Neumann passing the vertex with inner angle ω (see [7]).

Next, we consider the following interface problem

$$\begin{cases} -\nabla \cdot (\mathrm{a}\nabla u) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where the diffusion coefficient $\mathrm{a}(x)$ is a given piecewise constant function and f is a given function in $L^2(\Omega)$.

Solution of (1.1) has singular behavior near corners even when f is very smooth and so does the solution of (1.2). Such singular behavior affects the accuracy of the finite element method throughout the whole domain. Here and thereafter we use the standard notation and definition of the Sobolev spaces.

It is well-known that solution of (1.1) is in $H^{1+\alpha}(\Omega)$ with $\alpha < \frac{\pi}{2\omega}$ and that of (1.2) is in $H^{1+\alpha}(\Omega)$ with $\alpha > 0$ possibly close to zero, depending on the distribution of diffusion coefficients (see [13, 17]). Such low regularity of the solution makes lower order finite element approximations on quasi-uniform grids inaccurate. There were several approaches in the literatures for overcoming this difficulty, for example, local mesh refinement (see [1, 19]) and the p (or hp) version of the finite element method (see [20, 21]). Other approaches using the singular function decomposition of the solution include the singular function method (see, e.g., [11, 9]), the dual singular function method (see [10, 2, 3]), and the multigrid version of the dual singular function method (see [4]).

Recently new singular function(NSF) method was posed in [6, 8, 7, 16] for Poisson problem and interface problem. The approach is based on

the fact that solutions of such partial differential equations have singular function representations: a decomposition as the sum of regular and singular parts of the solution. Moreover, the singular part has an explicit form and the unknown coefficient of the singular part, the so-called *stress intensity factor*, is given by the *extraction formula* depending on the given data and the original solution. The singular and dual singular function methods augment approximation spaces by using the singular and/or dual singular functions. As an alternative, in this approach they develop, analyze, and test an accurate finite element method that also uses both the singular and dual singular functions. Differing from other methods of the singular function method, this method approximates the regular part of the solution that is much smoother than the solution itself. Once the regular part of the solution is computed, then the stress intensity factors and the solution itself can be calculated with negligible cost and without degrading accuracy. So the key of the method is to derive a well-posed and smoother problem for the regular part of the solution. They develop a new extraction formula for the stress intensity factors in terms of the regular part of the solution. The introduction, error analysis, and computations are in [6, 8, 7] for Poisson problems with corner singularities, and [16] for the interface problem.

We have explicit singular functions in the representations of the singular solutions for the Poisson problem and the interface problem. In this paper we show the Green's theorem is true for such singular functions. This observation leads us to consider the different type of weak problem from those in [6, 8, 16]. We find out this change gives poor approximation, by computation, and make an observation, which leads the conclusion that the choice of weak formulation of the original new singular function method is critical.

The paper is organized as follows. Section 2 introduces the singular functions for the Poisson problem with mixed boundary condition defined on a concave domain and two-dimensional interface problem, and

shows the singular functions satisfy the Green's theorem. Section 3 poses an alternative variational problem for the regular part using the result in Section 2. In section 4 we give numerical results, which show the variational form presented in Section 3 has poor convergence and give remarks that explain the results. Thus, we conclude that the variational form presented in [6, 8, 16] is the necessary choice.

2. Singular function representations

In this section we give the explicit forms of the singular functions which arise in the representations for two model problems and show that they satisfy the integration by parts. First we set

$$B(r_1; r_2) = \{(r, \theta) : \ r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1),$$

and define a family of cut-off functions of r, $\eta_\rho(r)$, as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\frac{1}{2}\rho R), \\ p(x) & \text{in } \bar{B}(\frac{1}{2}\rho R;\ \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases} \tag{2.1}$$

with $p(x) = \frac{15}{16}\left\{\frac{8}{15} - \left(\frac{4r}{\rho R} - 3\right) + \frac{2}{3}\left(\frac{4r}{\rho R} - 3\right)^3 - \frac{1}{5}\left(\frac{4r}{\rho R} - 3\right)^5\right\}$ where ρ is a parameter in $(0, 2]$ and $R \in \mathcal{R}$ is a fixed number which will be determined later so that the singular part $\eta_2 s$ has the same boundary condition as the solution u of the model problems. Here s is the singular function for the solution, whose explicit formula is given in subsection 2.1 and 2.2.

2.1. Singularities for Poisson equation

For the Poisson equation, the singular function depends on the boundary condition and the inner angle as in [7], where the explicit formulae

for singular functions are given for all possible cases. In this paper, we consider the (D/N) case with the inner angle ω, $\frac{3\pi}{2} < \omega < 2\pi$, so there are two singular functions of the form

$$s_1(r,\,\theta) = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega} \quad \text{and} \quad s_3(r,\,\theta) = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \tag{2.2}$$

and their corresponding dual singular functions

$$s_{-1}(r,\,\theta) = r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega} \quad \text{and} \quad s_{-3}(r,\,\theta) = r^{-\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}. \tag{2.3}$$

It is well-known that the solution of problem (1.1) has the following singular function representation:

$$u = w + \lambda_1\,\eta_\rho(r)s_1(r,\theta) + \lambda_3\,\eta_\rho(r)s_3(r,\theta), \tag{2.4}$$

where $w \in H^2(\Omega)\bigcap H^1_D(\Omega)$, with $H^1_D(\Omega) = \{v \in H^1(\Omega)\,:\, v = 0 \text{ on } \Gamma_D\}$, is the regular part of the solution and $\lambda_l \in \mathcal{R}\,(l = 1,3)$ are the stress intensity factors that can be expressed in terms of w by the following extraction formulas ([16]):

$$\lambda_l = \frac{1}{l\pi}(w,\,\Delta(\eta_2 s_{-l}))_{B(R;2R)} + \frac{1}{l\pi}(f,\eta_2 s_{-l})_{B(2R)}, \qquad l = 1,3. \tag{2.5}$$

Using (2.5) and substituting (2.4) into the Poisson equation (1.1), we obtain an integro-differential equation for w:

$$\begin{aligned} -\Delta w - \sum_{l=1,3} \frac{1}{l\pi}(w,\Delta(\eta_2 s_{-l}))_{B(R;2R)}\Delta(\eta_\rho s_l) \\ = f + \sum_{l=1,3} \frac{1}{l\pi}(f,\eta_2 s_{-l})_{B(2R)}\Delta(\eta_\rho s_l), \qquad \text{in } \Omega. \end{aligned}$$

Multiplying the above equation by a test function $v \in H^1_D(\Omega)$, integrating over Ω, and using integration by parts lead to the following variational problem:
Find $w \in H^2(\Omega)\bigcap H^1_D(\Omega)$ such that

$$a(w,v) = g(v) \quad \forall v \in H^1_D(\Omega), \tag{2.6}$$

where the bilinear form $a(\cdot,\cdot)$ and linear form $g(\cdot)$ are defined by

$$\begin{aligned} a(w,v) &= a^s(w,v) + b(w,v), \quad a^s(w,v) = (\nabla w, \nabla v), \\ b(w,v) &= -\sum_{l=1,3} \tfrac{1}{l\pi}\,(w, \Delta(\eta_2 s_{-l}))_{B(R;2R)} (\Delta(\eta_\rho s_l), v)_{B(\frac{1}{2}\rho R;\, \rho R)}, \end{aligned}$$

and

$$g(v) = (f,v) + \sum_{l=1,3} \frac{1}{l\pi} (f, \eta_2 s_{-l})_{B(2R)} (\Delta(\eta_\rho s_l), v)_{B(\frac{1}{2}\rho R;\, \rho R)},$$

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that $w \in H^2(\Omega)$ for $f \in L^2(\Omega)$. Note also that the bilinear form $a(\cdot,\cdot)$ is not symmetric.

2.2. Singularities for interface problem

In this subsection we give the formula of singular function using the standard process used in [16]. For simplicity of presentation, we assume that there is only one interface vertex p located at the origin. The general case with more reentrant corners and/or interface vertices can be also treated in a similar ways. It is well-known that the solution has a singular function representation (see [6, 14, 15, 17, 18]).

Let Ω_j (j = 1, ..., J) be open, polygonal subdomains of Ω and let $\{\Omega_j\}_{j=1}^J$ be a partition of the domain Ω

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \bigcup_{j=1}^{J} \bar{\Omega}_j = \bar{\Omega}.$$

Let $\Omega_\Sigma = \bigcup_{j=1}^J \Omega_j$. Assume that the diffusion coefficient a is piecewise constant with respect to the partition

$$\mathrm{a}(x) = \mathrm{a}_j > 0 \quad \text{in } \Omega_j \tag{2.7}$$

for j = 1, ... , J. Let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ denote the common edge of Ω_i and Ω_j and let $\mathbf{n}_j$ be the outward unit normal vector to the boundary $\partial\Omega_j$ of Ω_j. Then problem (1.2) can be rewritten as:
Find $u \in H_0^1(\Omega)$ such that

$$-\mathrm{a}_j\, \Delta u = f \quad \text{in } \Omega_j \tag{2.8}$$

for $j = 1, \dots, J$ with interface conditions

$$\mathrm{a}_i \frac{\partial u}{\partial \mathbf{n}_i}\bigg|_{\Gamma_{ij}} + \mathrm{a}_j \frac{\partial u}{\partial \mathbf{n}_j}\bigg|_{\Gamma_{ij}} = 0$$

for $i, j = 1, \dots, J$ such that $\Gamma_{ij} \neq \emptyset$ (see [14]).

Let $\Omega_{m_1}, \Omega_{m_2}, \dots, \Omega_{m_I}$ be the subdomains sharing p as a common vertex. Let $\delta > 0$ be a small number such that p is the only vertex of the subdomians inside the disc $D(p, \delta)$ centered at p with radius δ. When p belongs to the boundary of the domain Ω, let polar coordinates (r, θ) be chosen so that $D(p, \delta) \cap \Omega_{m_i} = \{(r, \theta) : 0 < r < \delta, \omega_{i-1} < \theta < \omega_i\}$ for $1 \leq i \leq I$, where $\omega_0 = 0$, and $\omega_I = \omega$ is the angle between the two edges of $\partial\Omega$ emanating from p. When p belongs to the interior of the domain Ω, the subdomians $\{\Omega_{m_i}\}_{i=1}^{I}$ completely surround p. So we may have the polar coordinates such that $\omega_0 = 0$ and $\omega_I = 2\pi$. Let $\lambda_k = (\alpha_k)^2$ and $\Theta_k(\theta)$ for $k \geq 1$ be, respectively, the positive eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville problem at the vertex: in subintervals (ω_{i-1}, ω_i) $(i = 1, \dots, I)$

$$-\Theta''(\theta) = \lambda\Theta(\theta),$$

on interfaces ω_i $(i = 1, \dots, I-1)$

(2.9)

$$\lim_{\theta\to\omega_i^-} \Theta(\theta) = \lim_{\theta\to\omega_i^+} \Theta(\theta) \quad \text{and} \quad \mathrm{a}_{m_i} \lim_{\theta\to\omega_i^-} \Theta'(\theta) = \mathrm{a}_{m_{i+1}} \lim_{\theta\to\omega_i^+} \Theta'(\theta),$$

and on boundaries $\theta = 0$, $\theta = \omega$ or 2π

$$\lim_{\theta\to 0^+} \Theta(\theta) = \lim_{\theta\to\omega^-} \Theta(\theta) = 0 \quad \text{if } p \in \partial\Omega,$$

$$\lim_{\theta\to 0^+} \Theta(\theta) = \lim_{\theta\to(2\pi)^-} \Theta(\theta) \quad \text{and} \quad \mathrm{a}_{m_1} \lim_{\theta\to 0^+} \Theta'(\theta) = \mathrm{a}_{m_I} \lim_{\theta\to(2\pi)^-} \Theta'(\theta) \quad \text{if } p \in \Omega,$$

where the eigenfunctions are normalized as follows

$$\sum_{i=1}^{I} \int_{\omega_{i-1}}^{\omega_i} \mathrm{a}_{m_i} \Theta_j(\theta)\Theta_k(\theta)\, d\theta = \delta_{jk} := \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Let $\alpha_1 \le \ldots \le \alpha_L$ be all $\alpha_l^{\prime}s$ that satisfy $0 < \alpha_l < 1$. Define the *singular functions* and the *dual singular functions* by

$$s_l(r,\theta) = r^{\alpha_l}\Theta_l(\theta) \quad \text{and} \quad s_{-l}(r,\theta) = r^{-\alpha_l}\Theta_l(\theta), \tag{2.10}$$

respectively. Note that s_l and s_{-l} are twice differentiable and harmonic in each subdomain Ω_j ($\Delta s_l = \Delta s_{-l} = 0$ in Ω_j). It is easy to see that for $i = 1, \ldots, I$

$$s_l \in H^{1+\alpha_l-\varepsilon}(\Omega_{m_i}) \quad \text{and} \quad s_{-l} \in H^{1-\alpha_l-\epsilon}(\Omega_{m_i})$$

for any $\varepsilon > 0$. On the interface $\Gamma_{m_i m_{i+1}} = \partial\Omega_{m_i} \cap \partial\Omega_{m_{i+1}}$, the second equation in (2.9) implies

$$\mathrm{a}_{m_i}\frac{\partial s_l}{\partial \mathbf{n}_{m_i}}\bigg|_{\Gamma_{m_i m_{i+1}}} + \mathrm{a}_{m_{i+1}}\frac{\partial s_l}{\partial \mathbf{n}_{m_{i+1}}}\bigg|_{\Gamma_{m_i m_{i+1}}} = 0,$$

$$\mathrm{a}_{m_i}\frac{\partial s_{-l}}{\partial \mathbf{n}_{m_i}}\bigg|_{\Gamma_{m_i m_{i+1}}} + \mathrm{a}_{m_{i+1}}\frac{\partial s_{-l}}{\partial \mathbf{n}_{m_{i+1}}}\bigg|_{\Gamma_{m_i m_{i+1}}} = 0.$$

Moreover, we have

$$\mathrm{a}_{m_i}\frac{\partial \eta_\rho s_l}{\partial \mathbf{n}_{m_i}}\bigg|_{\Gamma_{m_i m_{i+1}}} + \mathrm{a}_{m_{i+1}}\frac{\partial \eta_\rho s_l}{\partial \mathbf{n}_{m_{i+1}}}\bigg|_{\Gamma_{m_i m_{i+1}}} = 0, \tag{2.11}$$

$$\mathrm{a}_{m_i}\frac{\partial \eta_\rho s_{-l}}{\partial \mathbf{n}_{m_i}}\bigg|_{\Gamma_{m_i m_{i+1}}} + \mathrm{a}_{m_{i+1}}\frac{\partial \eta_\rho s_{-l}}{\partial \mathbf{n}_{m_{i+1}}}\bigg|_{\Gamma_{m_i m_{i+1}}} = 0.$$

Remark 2.1. *The singular functions s_l and the dual singular functions s_{-l}, both given in* (2.10)*, are defined in each subdomain Ω_j and infinitely differentiable there. However, their values and derivatives may not be defined on the interfaces $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, so we need to notify that some of the inner products or norms of functions containing such singular functions or their derivatives are the summation of their values on each subdomain Ω_j. For that reason we indicate the situation by using subindex Ω_Σ as follows:*

$$(\mathrm{a}\,\Delta\,(\eta_\rho\, s_k),\, \eta_2\, s_{-l})_{\Omega_\Sigma} = \sum_j (\mathrm{a}_j\,\Delta\,(\eta_\rho\, s_k),\, \eta_2\, s_{-l})_{\Omega_j}$$

and

$$\|\Delta\,(\eta_2\, s_{-l})\|^2_{\Omega_\Sigma} = \sum_j \|\Delta\,(\eta_2\, s_{-l})\|^2_{\Omega_j}.$$

Similarly, we use the subindex to emphasize the smaller supports in the norms and inner products as in $\|\Delta(\eta_2 s_{-l})\|_{B(R;2R)}$ *or* $(\mathrm{a}\phi, \Delta(\eta_2 s_{-l}))_{B(R;2R)}$ *in the section 3 and thereafter. Although we use these subindices to reduce the possible confusion or to emphasize the smaller supports but will omit or use only one of them to avoid possible confusion by the overuses, if necessary* ([5]).

Using the cut-off function defined above, the solution of (2.8) has the following singular function representation

$$u = w + \sum_{l=1}^{L} \kappa_l \eta_\rho s_l, \tag{2.12}$$

where $w \in H^2(\Omega_j)$ for $1 \le j \le J$ is the regular part of the solution and κ_l for $1 \le l \le L$ is the so-called stress intensity factors. The stress intensity factors can be expressed in terms of the following *extraction formula* (see, e.g., [5]):

$$\kappa_l = \frac{1}{2\alpha_l} \sum_{i=1}^{I} \int_{\Omega_{m_i}} [f \eta_\rho s_{-l} + \mathrm{a}_{m_i} u\, \Delta(\eta_\rho s_{-l})] dx, \tag{2.13}$$

where Ω_{m_i} are the subdomains sharing p as a common vertex.

In the NSF methods they use the following *extraction formula* whose integrand contains the regular part w of the solution as follows:

Lemma 2.1. *The stress intensity factors* κ_l *for* $1 \le l \le L$ *can be expressed in terms of* w *corresponding to* $\rho \le 1$ *by the following extraction formula*

$$\kappa_l = \frac{1}{2\alpha_l} \sum_{i=1}^{I} \int_{\Omega_{m_i}} [f \eta_2 s_{-l} + \mathrm{a}_{m_i} w\, \Delta(\eta_2 s_{-l})]\, dx, \tag{2.14}$$

where Ω_{m_i} *are the subdomains sharing* p *as a common vertex.*

Proof. See [16]. □

Now we give a modification of Green's theorem for the regular part of the solution.

Lemma 2.2. *Let $w \in H^2(\Omega_j)$ be the regular part of the solution in* (2.12). *Then*

$$\sum_j \int_{\Omega_j} \mathrm{a}_j \nabla w \cdot \nabla v dx = -\sum_j \int_{\Omega_j} (\nabla \cdot \mathrm{a}_j \nabla w)\, v dx \tag{2.15}$$

for any $v \in H_0^1(\Omega)$.

Proof. See [16]. □

Substituting the singular function representation of u in (2.12) into the model equation in (2.8), multiplying by a test function $v \in H_0^1(\Omega)$, integrating over domain Ω_Σ, and using Lemma 2.2 yield

$$\int_\Omega \mathrm{a}\nabla w \cdot \nabla v\, dx - \sum_{l=1}^{L} \kappa_l \int_{\Omega_\Sigma} \mathrm{a}\Delta(\eta_\rho s_l) \cdot v\, dx = \int_\Omega f v\, dx.$$

Using the extraction formula of κ_l in (2.14) and regrouping terms give

$$\begin{aligned}\int_\Omega \mathrm{a}\nabla w \cdot \nabla v dx \quad &- \quad \sum_{l=1}^{L} \frac{1}{2\alpha_l} \left(\sum_{i=1}^{I} \int_{\Omega_{m_i}} \mathrm{a}_{m_i} w\, \Delta(\eta_2 s_{-l})\, dx \right) \int_{\Omega_\Sigma} \mathrm{a}\Delta(\eta_\rho s_l) \cdot v\, dx \\ &= \int_\Omega f v\, dx + \sum_{l=1}^{L} \frac{1}{2\alpha_l} \left(\sum_{i=1}^{I} \int_{\Omega_{m_i}} f \eta_2 s_{-l}\, dx \right) \int_{\Omega_\Sigma} \mathrm{a}\Delta(\eta_\rho s_l) \cdot v\, dx.\end{aligned}$$

Then the variational problem for the regular part of the solution is to find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.16}$$

where the bilinear and linear forms are defined by

(2.17)

$$a(w, v) = (\mathrm{a}\nabla w, \nabla v) - \sum_{l=1}^{L} \frac{1}{2\alpha_l} (\mathrm{a}w\,, \Delta(\eta_2 s_{-l}))_{\Omega_\Sigma} (\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)_{\Omega_\Sigma}$$

and

$$g(v) = (f, v) + \sum_{l=1}^{L} \frac{1}{2\alpha_l} (f, \eta_2 s_{-l}) (\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)_{\Omega_\Sigma}, \tag{2.18}$$

respectively.

2.3. Green's theorems applied to singular functions

We use the following lemma from [12]:

Lemma 2.3. *Let $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Then $\frac{\partial u}{\partial \nu} \in H^{-\frac{1}{2}}(\partial\Omega)$ and*

$$\int_\Omega \Delta u \, v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, v ds$$

for any $v \in H^1(\Omega)$.

The following lemma shows that we may use the integration by parts, called as 'the Green's theorem', for the singular functions for the Poisson problem with domain singularities.

Lemma 2.4. *Let s_i be one of the singular functions in the singular representation* (2.4)*, then*

$$\int_\Omega \Delta \eta_\rho s_i \, v dx = -\int_\Omega \nabla \eta_\rho s_i \cdot \nabla v dx$$

for any $v \in H^1_D(\Omega)$.

Proof. Since $\eta_\rho = 1$ near the corner, we have $\Delta \eta_\rho s_i = \Delta s_i = 0$ there, which implies $\Delta \eta_\rho s_i \in L^2$. So, the lemma follows from Lemma 2.3, together with the boundary conditions of $\eta_\rho s_i$ and v. □

Now we have the similar result for the singular functions for two-dimensional interface problem as follows:

Lemma 2.5. *Let s_l be one of the singular functions in the singular representation* (2.12)*, then*

$$\sum_j \int_{\Omega_j} \mathrm{a}_j \Delta(\eta_\rho s_l) \, v dx = -\sum_j \int_{\Omega_j} \mathrm{a}_j \nabla \eta_\rho s_l \cdot \nabla v dx$$

for any $v \in H^1_0(\Omega)$.

Proof. Lemma 2.3, applied in each subdomain Ω_j, yields

$$\int_{\Omega_j} \mathrm{a}_j \, \Delta(\eta_\rho s_l) \, v dx + \int_{\Omega_j} \mathrm{a}_j \, \nabla(\eta_\rho s_l) \, \nabla v dx = \int_{\partial\Omega_j} \mathrm{a}_j \, \frac{\partial(\eta_\rho s_l)}{\partial \mathbf{n}_j} \, v ds,$$

and the summation gives the lemma, together with the boundary condition of v and the interface condition (2.11). □

Using the notation of inner product, we may express the lemma as the following;

$$(\mathrm{a} \, \Delta(\eta_\rho s_l) \, , v)_{\Omega_\Sigma} = -(\mathrm{a} \nabla(\eta_\rho s_l) \, , \nabla v)_{\Omega_\Sigma} \qquad \forall v \in H_0^1(\Omega).$$

3. Alternative variational problems for w

In this section we pose two alternative variational problems for w by applying the identities in Lemmas 2.4 and 2.5 to the righthand sides of problems: (2.6) and (2.16). First, for the Poisson problem with the mixed boundary condition, using the variational problem (2.6) and Lemma 2.4, we pose an alternative variational problem:
Find $w \in H^2(\Omega) \bigcap H_D^1(\Omega)$ such that

$$\bar{a}(w, v) = \bar{g}(v) \quad \forall v \in H_D^1(\Omega), \tag{3.1}$$

where the bilinear form $a(\cdot, \cdot)$ and linear form $g(\cdot)$ are defined by

$$\begin{aligned} \bar{a}(w, v) &= a^s(w, v) + \bar{b}(w, v), \quad a^s(w, v) = (\nabla w, \nabla v), \\ \bar{b}(w, v) &= \sum_{l=1,3} \tfrac{1}{l\pi} \, (w, \Delta(\eta_2 s_{-l}))_{B(R;2R)} (\nabla(\eta_\rho s_l), \nabla v)_{B(\frac{1}{2}\rho R;\, \rho R)}, \end{aligned}$$

and

$$\bar{g}(v) = (f, v) - \sum_{l=1,3} \frac{1}{l\pi} (f, \eta_2 s_{-l})_{B(2R)} (\nabla(\eta_\rho s_l), \nabla v)_{B(\frac{1}{2}\rho R;\, \rho R)}$$

Next, for the interface problem, using the variational problem (2.16) and Lemma 2.5, we pose an alternative variational problem:
Find $w \in H_0^1(\Omega)$ such that

$$\bar{a}(w, v) = \bar{g}(v) \quad \forall \, v \in H_0^1(\Omega), \tag{3.2}$$

where the bilinear and linear forms are defined by

$$\bar{a}(w,v) = (\mathrm{a}\nabla w, \nabla v) + \sum_{l=1}^{L} \frac{1}{2\alpha_l}(\mathrm{a}w\,, \Delta(\eta_2 s_{-l}))_{\Omega_\Sigma}(\mathrm{a}\,\nabla(\eta_\rho s_l),\, \nabla v)_{\Omega_\Sigma}$$

and

$$\bar{g}(v) = (f,v) - \sum_{l=1}^{L} \frac{1}{2\alpha_l}(f, \eta_2 s_{-l})(\mathrm{a}\,\nabla(\eta_\rho s_l),\, \nabla v)_{\Omega_\Sigma},$$

respectively.

4. Finite element space and numerical experiment

In this section we introduce the standard finite element space with examples and give computational results, which show that the alternative variational forms result in the poor convergence, and give conclusive notes in the final subsection.

4.1. Finite element space

We consider the standard finite space for the computational experiment. To this end, let $\mathcal{T}_h$ be a partition of the domain Ω into triangular finite elements, i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{\operatorname{diam} K\colon\ K \in \mathcal{T}_h\}$. Assume that any $K \in \mathcal{T}_h$ is a subset of Ω_j or $K \cap \Omega_j = \emptyset$ and the triangulation $\mathcal{T}_h$ is regular. Let V_h be continuous piecewise linear finite element space, i.e.,

$$V_h = \{\phi_h \in C^0(\Omega)\colon\ \phi_h|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h,\ \phi_h = 0 \text{ on } \partial\Omega\} \subset H_0^1(\Omega),$$

where $P_1(K)$ is the space of linear functions on K.

4.2. Numerical examples and computational results

In this subsection we have computational results for three examples. First we have the computational results for NSF method applied to the interface problem on a gamma shaped domain; the results show that

the finite approximation made from the variational problem (2.16) gives optimal in H^1-norm and compatible results in L^2-norm([16]).

Then we consider two examples, for which we applied two numerical experiments made from two different variational problems, (2.16) and (3.2). One example is the Poisson problem made from first example by assuming $\mathrm{a} \equiv 1$. The another is the same as the first one with a simplified input function f.

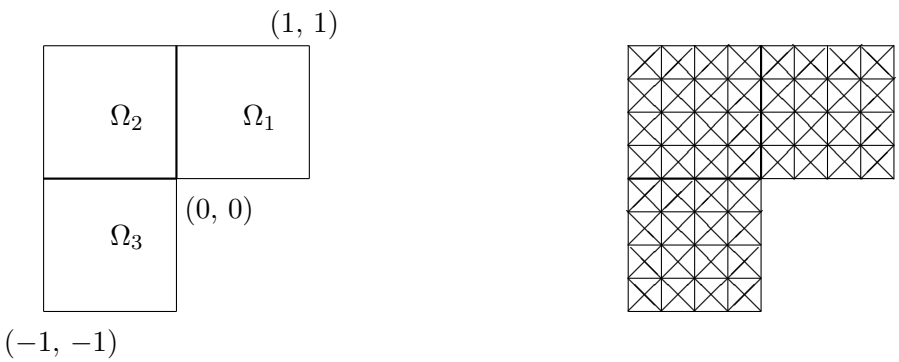

FIGURE 1. A Γ-shaped domain partitioned into three squares and its partitions.

Example 1 : The first numerical test is on an interface problem with a corner singular point. Consider the Γ-shaped domain with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(0,-1)$, $(0,0)$, and $(1,0)$, and partition the domain into three squares Ω_1, Ω_2, and Ω_3 as depicted in Fig.1. Let the diffusion coefficient $\mathrm{a}(x)$ be piecewise constant, i.e., $\mathrm{a}(x) = \mathrm{a}_i$ on Ω_i, where

$$\mathrm{a}_1 = \mathrm{a}_3 = 1 \quad \text{and} \quad \mathrm{a}_2 = 100. \tag{4.1}$$

Then the corresponding interface problem (1.2) has only one interface vertex at the origin, and its only singular function has the form of

$$s = r^{\alpha}\,\Theta(\theta) \tag{4.2}$$

with $\alpha = 0.089658901772145...$ and $\Theta(\theta) = C_i \sin(\alpha\theta) + D_i \cos(\alpha\theta)$ on Ω_i for $i = 1, 2, 3$. Let η_2 be the cut-off function defined in Section 2 with $R = 1/8$. Choose the right-hand side function in (2.8) to be

$$f = -\mathrm{a}_i \left(-\frac{6}{\mathrm{a}_i}\, x(y^2 - y^4) + \frac{1}{\mathrm{a}_i}(x - x^3)(2 - 12y^2) + \Delta(\eta_2 s) \right) \quad \text{on } \Omega_i$$

so that the exact solution is

$$u = w_\rho + \eta_\rho s,$$

where η_ρ is the cut-off function with $R = 1/8$ and $0 < \rho \le 1$ and w_ρ is the regular part of the solution having the form of

$$w_\rho = \frac{1}{\mathrm{a}_i}(x - x^3)(y^2 - y^4) + (\eta_2 - \eta_\rho)s \quad \text{on } \Omega_i$$

for $i = 1, 2, 3$.

Tables 1 reports numerical results of the discretization error of the regular part of the solution in the respective L^2, L^∞, and H^1- norms for $\rho = 1$. Even though the errors change with respect to various values of ρ, the difference is insignificant (see [16]).

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
L^2	5.0549e-03	1.8321e-03	4.8524e-04	1.1252e-04	2.1761e-05	5.84033e-06
	Order	1.4642	1.9167	2.1085	2.3704	1.897623
L^∞	3.2061e-02	1.9981e-02	1.4292e-03	1.2309e-03	1.5529e-04	6.32335e-05
	Order	0.6822	3.8053	0.2156	2.9866	1.296204
H^1	1.2451e-01	6.1012e-02	1.7750e-02	9.1545e-03	4.2923e-03	2.1391e-03
	Order	1.0291	1.7812	0.9552	1.0927	1.004734
κ	1.8632	1.3415	1.0399	1.0207	1.0035	1.00111
	Order	1.337809	3.0097425	0.946758	2.564204	1.656795

TABLE 1. Discretization Error for $\rho = 1.0$

Now we give second example with two numerical experiments whose finite approximations are made from two different variational problems (2.16) and (3.2), respectively. We note the term $(\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)$ in the bilinear and linear forms in the respective (2.17) and (2.18) was replaced by $-(\mathrm{a}\,\nabla(\eta_\rho s_l),\, \nabla v)$.

Example 2 : Consider the Poisson problem (1.1) defined on the Γ-shaped domain. We note (1.2) becomes the Poisson problem if we assume $\mathrm{a} \equiv 1$ in the first example. Moreover, we use more simplified input function f. Let η_2 be defined in (2.1) with $R = \frac{1}{8}$ and let $f = -\Delta\eta_2 s$. Then the exact solution is

$$u = \eta_2 s = (\eta_2 - \eta_\rho)s + \eta_\rho s \equiv w_\rho + \kappa\eta_\rho s,$$

where $s = r^{\frac{2}{3}}\sin(\frac{2}{3}\,\theta) \in H^{5/3-\epsilon}(\Omega)$ for any $\epsilon > 0$ and $\kappa = 1$. Note that the solution of this example is in $H^{5/3-\epsilon}(\Omega)$ which is reasonably smooth. For $\rho = 1$, Tables 2 and 3 depict the discretization error based on variational problems (2.16) and (3.2), respectively. It is clear that both the methods are $O(h)$ accurate in the H^1- norm, but approximation based on (3.2) is not optimal in the L^2- norm and is less accurate than that of (2.16).

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
L^2	1.1649e-03	2.8585e-04	7.2031e-05	1.7985e-05	4.5048e-06	1.1276e-06
	Order	2.026928	1.988563	2.001776	1.997295	1.998121
L^∞	6.5731e-03	1.5884e-03	4.2078e-04	1.0937e-04	2.6755e-05	6.6931e-06
	Order	2.048963	1.916480	1.943741	2.031407	1.999096
H^1	1.5219e-01	7.5056e-02	3.7816e-02	1.8957e-02	9.4851e-03	4.7433e-03
	Order	1.019862	0.988976	0.996217	0.999048	0.999758

TABLE 2. Discretization Error based on (2.16)

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
L^2	1.1485e-03	3.2791e-04	1.0252e-04	3.4343e-05	1.2576e-05	4.8117e-06
	Order	1.808471	1.677348	1.577826	1.449354	1.386053
L^∞	4.9299e-03	3.1377e-03	1.9790e-03	1.2469e-03	7.8566e-04	4.9496e-04
	Order	0.651836	0.664968	0.666368	0.666427	0.666573
H^1	1.5803e-01	7.8354e-02	3.9671e-02	1.9955e-02	1.0025e-02	5.0439e-03
	Order	1.012182	0.981944	0.991273	0.993170	0.991017

TABLE 3. Discretization Error based on (3.2)

The final example considered is the same interface problem as in the first example with a simplified input function f as in Example 2.

Example 3 : Let $f = -\mathrm{a}\Delta\eta_2 s$ with a and s defined in as in (4.1) and (4.2), respectively. Then the exact solution is

$$u = \eta_2 s = (\eta_2 - \eta_\rho)s + \eta_\rho s \equiv w_\rho + \kappa\eta_\rho s \in H^{1+\alpha}$$

with $\kappa = 1$, $\alpha < 0.09$, and $\rho = 1$. Numerical results based on problem (2.16) in Table 4 are similar to those for the first example. Discretization error depicted in Table 5 is for approximation based on problem (3.2). Clearly, it is much worse than those in Table 4 even though it does converge.

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
L^2	1.0054e-03	1.8551e-03	3.0808e-04	8.9421e-05	7.1347e-06	2.3604e-06
	Order	-0.883704	2.590083	1.784621	3.647698	1.595801
L^∞	8.7340e-03	6.7284e-03	1.2722e-03	3.5363e-04	4.9619e-05	1.2991e-05
	Order	0.37637	2.402894	1.847041	2.833284	1.933323
H^1	7.4352e-02	3.5576e-02	1.6883e-02	8.3786e-03	4.1771e-03	2.0878e-03
	Order	1.063469	1.075256	1.010867	1.004204	1.000466

TABLE 4. Discretization Error based on (2.16)

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
L^2	5.7852e-03	6.3918e-03	5.0587e-03	4.3433e-03	3.6812e-03	3.1454e-03
	Order	-0.143866	0.337451	0.219978	0.238597	0.226959
L^∞	3.3067e-02	3.3963e-02	3.2641e-02	3.1468e-02	3.0131e-02	2.8751e-02
	Order	-0.038567	0.057260	0.052798	0.062683	0.067647
H^1	9.6776e-02	7.8118e-02	7.0617e-02	6.8220e-02	6.6411e-02	6.4671e-02
	Order	0.308995	0.145641	0.049814	0.038783	0.038290

TABLE 5. Discretization Error based on (3.2)

4.3. Conclusive notes

In Sections 2 and 3 we showed that the Green's theorem is true for the singular functions which arise in the Poisson problems and interface problems and posed another variational problems by replacing the term $(\Delta(\eta_\rho s_l),\, v)$ by $-(\nabla(\eta_\rho s_l), \nabla v)$ and $(\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)$ by $-(\mathrm{a}\,\nabla(\eta_\rho s_l),\, \nabla v)$, respectively.

Noting the similarities of the statements between the two cases, we only use the interface case to give the conclusive notes.

Recall the original variational problem for the regular part of the solution is to find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall\, v \in H_0^1(\Omega), \tag{4.3}$$

where the bilinear and linear forms are defined by

$$a(w, v) = (\mathrm{a}\nabla w, \nabla v) - \sum_{l=1}^{L} \frac{1}{2\alpha_l}(\mathrm{a}w\,, \Delta(\eta_2 s_{-l}))_{\Omega_\Sigma}(\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)_{\Omega_\Sigma} \tag{4.4}$$

and

$$g(v) = (f, v) + \sum_{l=1}^{L} \frac{1}{2\alpha_l}(f, \eta_2 s_{-l})(\mathrm{a}\,\Delta(\eta_\rho s_l),\, v)_{\Omega_\Sigma}, \tag{4.5}$$

respectively.

Next, our new variational problem motivated by the Lemma 2.5 is as follows:

Find $w \in H_0^1(\Omega)$ such that

$$\bar{a}(w, v) = \bar{g}(v) \quad \forall\, v \in H_0^1(\Omega), \tag{4.6}$$

where the bilinear and linear forms are defined by

(4.7)

$$\bar{a}(w, v) = (\mathrm{a}\nabla w, \nabla v) + \sum_{l=1}^{L} \frac{1}{2\alpha_l}(\mathrm{a}w\,, \Delta(\eta_2 s_{-l}))_{\Omega_\Sigma}(\mathrm{a}\,\nabla(\eta_\rho s_l),\ \nabla v)_{\Omega_\Sigma}$$

and

$$\bar{g}(v) = (f, v) - \sum_{l=1}^{L} \frac{1}{2\alpha_l}(f, \eta_2 s_{-l})(\mathrm{a}\,\nabla(\eta_\rho s_l),\ \nabla v)_{\Omega_\Sigma}, \tag{4.8}$$

respectively.

First we observe that we have $(\mathrm{a}\,\Delta(\eta_\rho s_l)\,, v)_{\Omega_\Sigma} = -(\mathrm{a}\nabla(\eta_\rho s_l)\,, \nabla v)_{\Omega_\Sigma}$ for all $v \in H_0^1(\Omega)$ and even for $v \in V_h$. So the variational problems (4.3) and (4.6) give the same weak solution and their finite approximations of them give the same limit. But we note that when we compute approximated value of $(\mathrm{a}\nabla(\eta_\rho s_l), \nabla v)_{\Omega_\Sigma}$ in V_h, we only have an approximated value with poor convergence. In fact, the finite approximation for the problem

$$(\mathrm{a}\nabla w\,, \nabla v)_{\Omega_\Sigma} = (g\,, v)_{\Omega_\Sigma} \qquad \forall v \in V_h,$$

with $g = -\mathrm{a}\,\Delta(\eta_\rho s_l) \in L^2$ has only $O(h^\alpha)$ convergence in H^1-norm, as a standard FEM applied to a problem with a domain singularity.

Here we summarize the conclusions :

Note 1. Since $(\mathrm{a}\,\Delta(\eta_\rho s_l)\,, v)_{\Omega_\Sigma} = -(\mathrm{a}\nabla(\eta_\rho s_l)\,, \nabla v)_{\Omega_\Sigma}$ for all $v \in H_0^1(\Omega)$, the variational problems (4.3) and (4.6) give the same weak solution.

Note 2. The finite element approximations for both variational problems have the same limit.

Note 3. Computing the approximated value $-(\mathrm{a}\nabla(\eta_\rho s_l)\,, \nabla v)_{\Omega_\Sigma}$ in V_h instead of $(\mathrm{a}\Delta(\eta_\rho s_l)\,, v)_{\Omega_\Sigma}$ brings the poor convergence to the alternative variational problem.

Note 4. In conclusion, we should use the finite element approximation made from the weak problem (4.3) to get the optimal convergence.

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