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Dynamics of the g-Navier–Stokes equations

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Abstract

The 2D g-Navier–Stokes equations has the following form:

$$\frac{\partial \mathbf{u}}{\partial t} - v\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}$$

with the continuity equation

$$\nabla \cdot (g\mathbf{u}) = 0,$$

where g is a suitable smooth real-valued function. For the restricted function g, Roh showed the existence of the global attractors for the periodic boundary conditions. One note that we get the 2D Navier–Stokes equations for $g = 1$.

Therefore, in this paper we are interested in the behavior of the global attractors of the 2D g-Navier–Stokes equations as $g \to 1$ in the proper sense and will prove that the semiflows, generated by the projection of the solutions of the g-Navier–Stokes equations into the solution space of the Navier–Stokes equations, is robust at the global attractor of the Navier–Stokes equations with respect to g.

For that, we will use the Robustness theorem developed by Sell and You.

Keywords: g-Navier–Stokes equations; Weak solution; Strong solution; Attractor; Robustness of global attractors

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1. Introduction

In this paper, we study the behavior of solutions of the g-Navier–Stokes equations in spatial dimension 2. These equations are a variation of the standard Navier–Stokes equations, and they assume the form,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{1}{g}(\nabla\cdot g\mathbf{u}) = \frac{\nabla g}{g}\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on $(x_1, x_2) \in \Omega$ and Ω is a suitable bounded domain in $\mathbf{R}^2$. Notice that if $g(x_1, x_2) = 1$, then the Eqs. (1) and (2) reduce to the standard Navier–Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{4}$$

Of particular interest in this paper, is the problem where the gradient ∇g is small and g is "close to" 1. In this case, one can view the g-Navier–Stokes equations as a small perturbation of the standard Navier–Stokes equations. We are interested in comparing the long-time dynamics of the solution of the two systems.

While the g-Navier–Stokes equations form a meaningful problem in a 3D spatial region $\Omega \subset \mathbf{R}^3$, where $g = g(x_1, x_2, x_3)$ and $(x_1, x_2, x_3) \in \Omega$, we are specially interested in the 2D problem here. The reason for this is that the 2D g-Navier–Stokes equations arise in a natural way in the study of a standard 3D problem, as we show in the next section. We do not claim that the g-Navier–Stokes equations form a model of any fluid flow. They may, or may not. That they are derived from a standard 3D problem is the basis for our study.

Before we present the derivation of the g-Navier–Stokes equations, it is convenient to recall some relevant aspects of the classical theory of the Navier–Stokes equations. For many years, the Navier–Stokes equations were investigated by many authors and the existence of the attractors for 2D Navier–Stokes equations was first proved by Ladyzhenskaya [3] and independently by Foias and Temam [2]. The finite-dimensional property of the global attractor for general dissipative equations was first proved by Mallet–Paret [5] and Mañé [6]. For the analysis on the Navier–Stokes equations, one can refer to [1,4,8,9], specially [10] for the periodic boundary conditions.

In this paper, we will have the following organization. In Section 2, we will present the derivation of 2D g-Navier–Stokes equations from 3D Navier–Stokes equations without the proofs (see Roh [7] for the details). For the boundary conditions, we will consider the periodic boundary conditions, while we can get same results for the Dirichlet boundary conditions on the smooth bounded domain. In Section 3, we will present the mathematical spaces and the preliminary results of the g-Navier–Stokes equations.

Finally, in Section 4 we will prove the robustness of the global attractors with respect to the function g in the proper space at $g = 1$.

2. Derivation of the 2D g-Navier–Stokes equations

Let $\Omega_g = \Omega_2 \times [0, g]$, where Ω_2 is a bounded region in the plane and $g = g(x_1, x_2)$ is a smooth function defined on Ω_2 with $0 < m \leqslant g(x_1, x_2) \leqslant M$, for $(x_1, x_2) \in \Omega_2$. Now, we consider the 3D Navier–Stokes equations,

$$\frac{\partial \mathbf{U}}{\partial t} - \nu \Delta \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla \Phi = \mathbf{F} \quad \text{in} \quad \Omega_g,$$
$$\nabla \cdot \mathbf{U} = 0 \quad \text{in} \quad \Omega_g$$

with the boundary condition

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial_{\text{top}} \Omega_g \cup \partial_{\text{bottom}} \Omega_g, \tag{5}$$

where

$$\begin{aligned} \partial_{\text{top}} \Omega_g &= \{(x_1, x_2, x_3) \in \Omega_g : x_3 = g(x_1, x_2)\}, \\ \partial_{\text{bottom}} \Omega_g &= \{(x_1, x_2, x_3) \in \Omega_g : x_3 = 0\}. \end{aligned}$$

The lateral boundary condition corresponding to $\partial \Omega_2$ does not affect to the derivation of the 2D g-Navier–Stokes equations. But, in this paper we will consider the periodic boundary conditions to study the 2D g-Navier–Stokes equations.

Now we define $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ as

$$\mathbf{u}_i = \mathbf{u}_i(x_1, x_2) = \frac{1}{g(x_1, x_2)} \int_0^{g(x_1, x_2)} \mathbf{U}_i(x_1, x_2, x_3)\, dx_3$$

for $i = 1, 2$ and we get the following proposition.

Proposition 1. *Assume that* $\nabla \cdot \mathbf{U} = 0$ *in* Ω_g *and that* (5) *is valid. Then one has*

$$\nabla_2 \cdot (g\mathbf{u}) = \frac{\partial (g\mathbf{u}_1)}{\partial x_1} + \frac{\partial (g\mathbf{u}_2)}{\partial x_2} = \nabla g \cdot \mathbf{u} + g(\nabla_2 \cdot \mathbf{u}) = 0 \quad in\ \Omega_2,$$

where $\nabla_2 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ *and* $\nabla g = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right)$.

Proof. See Roh [7] for the details. □

Now, we consider the special case like

$$\mathbf{U}(x_1, x_2, x_3) = (\mathbf{U}_1(x_1, x_2), \mathbf{U}_2(x_1, x_2), \mathbf{U}_3(x_1, x_2, x_3)).$$

By the previous proposition, for $\mathbf{u} = (U_1, U_2)$, one has $\nabla \cdot (g\mathbf{u}) = 0$ and $\mathbf{u}$ satisfies the 2D g-Navier–Stokes equations. Moreover, we have

$$\mathbf{U}_3(x_1, x_2, x_3) = -x_3 \left(\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2} \right) = -x_3(\nabla_2 \cdot \mathbf{u}),$$

when (5) and $\nabla \cdot \mathbf{U} = 0$ in Ω_g are valid. This is the basis for our study of the 2D g-Navier–Stokes equations.

3. Preliminaries

In this section, we will present the results of the g-Navier–Stokes equations one can find in Roh [7]. Here, we consider the periodic boundary conditions on the domain $\Omega = (0, 1) \times (0, 1)$ and assume $\mathbf{u}, p$ and the first derivatives of $\mathbf{u}$ to be spatially periodic, i.e.,

$$\mathbf{u}(x_1 + 1, x_2) = \mathbf{u}(x_1, x_2) = \mathbf{u}(x_1, x_2 + 1), \quad (x_1, x_2) \in \mathbf{R}^2$$

and similarly for p and $\frac{\partial \mathbf{u}_i}{\partial x_j}$.

For the function g, throughout this paper, we assume that

1. $g(\mathbf{x}) \in C^\infty_{\text{per}}(\Omega)$ and
2. $0 < m \leqslant g(x, y) \leqslant M$ for all $(x, y) \in \Omega$.

Note that the constant function $g = 1$ is also included for our function g.

Now, we define the Hilbert space $L^2(\Omega, g) = L^2(\Omega, R^2, g)$, which is the space $L^2(\Omega)$ with the scalar product and the norm given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_g = \int_\Omega (\mathbf{u} \cdot \mathbf{v}) \; g \, d\mathbf{x} \quad \text{and} \quad \| \mathbf{u} \|_g^2 = \langle \mathbf{u}, \mathbf{u} \rangle_g,$$

where $\mathbf{x} = (x_1, x_2)$. Similarly, we define the space $H^1(\Omega, g)$ which is the space $H^1(\Omega)$ with the norm by

$$\| \mathbf{u} \|_{H^1(\Omega,g)} = \left[\langle \mathbf{u}, \mathbf{u} \rangle_g + \sum_{i=1}^{2} \langle D_i \mathbf{u}, D_i \mathbf{u} \rangle_g \right]^{\frac{1}{2}},$$

where $\frac{\partial \mathbf{u}}{\partial x_i} = D_i\mathbf{u}$. Specially, for the constant function $g = 1$, we denote that

$$\| \mathbf{u} \|_1 = \| \mathbf{u} \|, \quad \| \mathbf{u} \|_{H^1(\Omega,1)} = \| \mathbf{u} \|_{H^1(\Omega)}.$$

One can see easily that the norm $\| \mathbf{u} \|$ is equivalent to the norm $\| \mathbf{u} \|_g$ as well as the norm $\| \mathbf{u} \|_{H^1(\Omega)}$ is equivalent to the norm $\| \mathbf{u} \|_{H^1(\Omega,g)}$.

Now, we consider the following closed subspaces of $L^2(\Omega, g)$:

$$\tilde{H} = CL_{L^2(\Omega,g)}\{\mathbf{u} \in C^\infty_{\text{per}}(\Omega) \ : \ \nabla \cdot g\mathbf{u} = 0\}.$$

Then, we define the orthogonal projection $\tilde{P} : L^2_{\text{per}}(\Omega, g) \mapsto \tilde{H}$ and we can get $Q = \tilde{H}^\perp$ as

$$Q = CL_{L^2(\Omega)}\{\nabla\phi : \phi \in C^1_{\text{per}}(\bar{\Omega}, R)\},$$

which do not depend on the function g.

Therefore, for the given $\mathbf{v} \in L^2_{\text{per}}(\Omega, g)$, we can find $\mathbf{u} \in \tilde{H}$ and $\nabla p \in Q$ such that $\mathbf{v} = \mathbf{u} + \nabla p$.

But, for our problem, we are interested in the dynamics on the following spaces:

$$H_g = CL_{L^2(\Omega,g)}\left\{\mathbf{u} \in C^\infty_{\text{per}}(\Omega) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\, d\mathbf{x} = 0\right\},$$

$$V_g = \left\{\mathbf{u} \in H^1_{\text{per}}(\Omega, g) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\, d\mathbf{x} = 0\right\},$$

where H_g is endowed with the scalar product and the norm in $L^2(\Omega, g)$, and V_g is the spaces with the scalar product and the norm given by

$$\langle \mathbf{u}, \mathbf{v}\rangle_{V_g} = \int_\Omega (D_i\mathbf{u} \cdot D_i\mathbf{v})\ g\, d\mathbf{x} \quad \text{and} \quad \| \mathbf{u} \|^2_{V_g} = \langle \mathbf{u}, \mathbf{u}\rangle_{V_g}, \tag{6}$$

where $\mathbf{x} = (x_1, x_2)$.

Also, for a given $\mathbf{v} \in L^2_{\text{per}}(\Omega, g)$, one obtains

$$\mathbf{v} = \mathbf{u} + \frac{\mathbf{k}}{g} + \nabla p \quad \text{for } \mathbf{u} \in H_g, \quad \nabla p \in Q, \quad \mathbf{k} = \frac{1}{\int_\Omega \frac{1}{g}\, d\mathbf{x}} \int_\Omega \mathbf{v}\, d\mathbf{x} \tag{7}$$

and specially for $g = 1$ one has

$$\mathbf{v} = \mathbf{u} + \mathbf{k} + \nabla p \quad \text{for } \mathbf{u} \in H_1, \quad \nabla p \in Q, \quad \mathbf{k} = \int_\Omega \mathbf{v}\, d\mathbf{x}. \tag{8}$$

As a result, we can define the orthogonal projection $P_g : L^2_{\mathrm{per}}(\Omega, g) \mapsto H_g$, which is similar to the Lerary projection, as $P_g\mathbf{v} = \mathbf{u}$.

Now, throughout this paper we define the g-Laplacian Δ_g by

$$-\Delta_g \mathbf{u} = -\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{u} = -\Delta\mathbf{u} - \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u},$$

which is a perturbation of $-\Delta\mathbf{u}$. Then, for $\nu = 1$, (1) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta_g \mathbf{u} + \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega. \tag{9}$$

Thus, by taking the orthogonal projection P_g into (9), one obtains

$$\frac{d\mathbf{u}}{dt} + A_g\mathbf{u} + B_g(\mathbf{u}, \mathbf{u}) = \mathbf{q} \quad \text{on} \quad H_g, \tag{10}$$

where $A_g\mathbf{u} = P_g(-\Delta_g\mathbf{u})$, $B_g(\mathbf{u}, \mathbf{u}) = P_g(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\mathbf{q} = P_g[\mathbf{f} - \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u}]$. In this paper, we will call the linear operator $A_g\mathbf{u} = P_g(-\Delta_g\mathbf{u})$ as the g-Stokes operator. Also, we obtain the following proposition for the g-Stokes operator.

Proposition 2. *For the g-Stokes operator A_g, the followings hold*:

(i) *The g-Stokes operator A_g is a positive, self-adjoint operator with compact inverse, where the domain of A_g, $\mathcal{D}(A_g) = V_g \cap H^2(\Omega, g)$.*

(ii) *There exist countable eigenvalues of A_g satisfying*

$$0 < \lambda(g) \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots,$$

where $\lambda(g) = \frac{4\pi^2 m}{M}$ and λ_1 is the smallest eigenvalue of A_g. In addition, there exist the corresponding collection of eigenfunctions $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$ forms an orthonormal basis for H_g.

Then, for the fractional power of the g-Stokes operator, one can obtain same results as the one in Sell and You [8]. Since the operator A_g is self-adjoint one can have

$$\langle A_g\mathbf{u}, \mathbf{u}\rangle_g = \langle A_g^{\frac{1}{2}}\mathbf{u}, A_g^{\frac{1}{2}}\mathbf{u}\rangle_g \quad \text{for } \mathbf{u} \in \mathcal{D}(A_g) = V_g \cap H^2(\Omega, g) \tag{11}$$

and since the orthogonal projection P_g is self-adjoint operator, by using integration by parts we have

$$\langle A_g^{\frac{1}{2}}\mathbf{u}, A_g^{\frac{1}{2}}\mathbf{u}\rangle_g = \langle A_g\mathbf{u}, \mathbf{u}\rangle_g = \left\langle P_g\left[-\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{u}\right], \mathbf{u}\right\rangle_g = \int_\Omega (\nabla\mathbf{u} \cdot \nabla\mathbf{u})g\, d\mathbf{x},$$

which implies by (6) that

$$\| A_g^{\frac{1}{2}}\mathbf{u} \|_g^2 = \| \nabla \mathbf{u} \|_g^2 = \| \mathbf{u} \|_{V_g}^2 \quad \text{for} \quad \mathbf{u} \in V_g. \tag{12}$$

In addition, for $\mathbf{u} \in \mathcal{D}(A_g^{\alpha})$ and $0 \leqslant \alpha \leqslant 1$, one specially obtains

$$\lambda_1^{2\alpha} \| \mathbf{u} \|_g^2 \leqslant \| A_g^{\alpha}\mathbf{u} \|_g^2 \quad \text{and} \quad \| \mathbf{u} \|_{H^{2\alpha}(\Omega,g)} \leqslant \tilde{\delta} \| A_g^{\alpha}\mathbf{u} \|_g \tag{13}$$

for some positive $\tilde{\delta} = \tilde{\delta}(\alpha, \lambda_1)$, where λ_1 is the first eigenvalue of A_g.

Next, we denote the bilinear operator $B_g(\mathbf{u}, \mathbf{v}) = P_g(\mathbf{u} \cdot \nabla)\mathbf{v}$ and the trilinear form

$$b_g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_{\Omega} \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{w}_j g \, dx,$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lie in appropriate subspaces of $L^2_{\text{per}}(\Omega, g)$ and $D_i = \frac{\partial}{\partial x_i}$.

Then, one obtains

$$\begin{aligned} b_g(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \sum_{i,j=1}^{2} \int_{\Omega} \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{w}_j g \, dx = \sum_{i,j=1}^{2} \int_{\Omega} g \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{w}_j \, dx \\ &= -\sum_{i,j=1}^{2} \int_{\Omega} D_i (g \mathbf{u}_i) \mathbf{v}_j \mathbf{w}_j \, dx - \sum_{i,j=1}^{2} \int_{\Omega} g \mathbf{u}_i \mathbf{v}_j (D_i \mathbf{w}_j) \, dx \\ &= -\sum_{i,j=1}^{2} \int_{\Omega} g \mathbf{u}_i \mathbf{v}_j (D_i \mathbf{w}_j) \, dx = -b_g(\mathbf{u}, \mathbf{w}, \mathbf{v}) \end{aligned}$$

for sufficient smooth functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H_g$ and hence $b_g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b_g(\mathbf{u}, \mathbf{w}, \mathbf{v})$ which implies $b_g(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$. For the nonlinear term we obtained same results as the one in Section 6.1.2 of Sell and You [8].

Proposition 3. *Let $\alpha_i, i = 1, 2, 3$ be nonnegative real numbers that satisfy*

$$\alpha_1 + \alpha_2 + \alpha_3 \geqslant 1$$

and the vector $(\alpha_1, \alpha_2, \alpha_3)$ is not equal to $(1, 0, 0)$, nor $(0, 1, 0)$, nor $(0, 0, 1)$. Then there are positive constants $\gamma_i = \gamma_i(g, \alpha_1, \alpha_2, \alpha_3, \Omega)$ for $i = 1, 2$ such that

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant \gamma_1 \| \mathbf{u} \|_{H^{\alpha_1}} \| \mathbf{v} \|_{H^{(\alpha_2+1)}} \| \mathbf{w} \|_{H^{\alpha_3}},$$

where $\mathbf{u} \in H^{\alpha_1}$, $\mathbf{v} \in H^{\alpha_2+1}$ *and* $\mathbf{w} \in H^{\alpha_3}$, *and*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leqslant \gamma_2 \, \| A_g^{\frac{\alpha_1}{2}} \mathbf{u} \|_g \| A_g^{\frac{(\alpha_2+1)}{2}} \mathbf{v} \|_g \| A_g^{\frac{\alpha_3}{2}} \mathbf{w} \|_g$$

for all $\mathbf{u} \in V_g^{\alpha_1}$, $\mathbf{v} \in V_g^{(\alpha_2+1)}$ *and* $\mathbf{w} \in V_g^{\alpha_3}$.

Now, we are in the position to see the existence of the solutions of the g-Navier–Stokes equations. For the proofs, Roh [7] followed the presentation given by Sell and You [8].

Proposition 4. *Let* $\mathbf{f} \in L^\infty(0,\infty; L^2(\Omega, g))$ *be given. Then for every* $\mathbf{u}_0 \in H_g$ *there is precisely one weak solution* (*of class LH*) $\mathbf{u} = \mathbf{u}(t)$ *on* $[0,\infty)$ *of* (10), *satisfying* $\mathbf{u}(0) = \mathbf{u}_0$. *Moreover one has* $\mathbf{u}(t) \in C[0,\infty; H_g)$. *Also, let* $\mathbf{u} = \mathbf{u}(t)$ *be any weak solution of* (10) *on* $[0,\infty)$ *with initial condition* $\mathbf{u}(0) = \mathbf{u}_0 \in H_g$. *Then for each* $t_0 > 0$, $\mathbf{v}(t) = \mathbf{u}(t+t_0)$ *is a strong solution of* (10) *on* $[0,\infty)$ *with initial condition* $\mathbf{v}(0) = \mathbf{u}(t_0)$ *and* $D_t\mathbf{u} \in L^2_{\text{loc}}(0,\infty; H_g)$.

In next theorem, we will see uniqueness and continuity of the solution with respect of the data $\mathbf{u}_0$, $\mathbf{f}$.

Proposition 5. *Let* $\mathbf{f}_i \in L^\infty[0,\infty; L^2(\Omega, g))$ *be given and* d_i, $i = 1,\ldots,5$, *are constants. For* $i = 1, 2$, *let* $\mathbf{u}_i = \mathbf{u}_i(t)$ *denote two solutions of* (10) *in Proposition* 4 *defined on the interval* $[0,\infty)$, *with data* $(\mathbf{u}_i(0), \mathbf{f}_i)$. *We also denote* $\rho_1(t)$, $\rho_2(t)$ *and* $\rho_3(t)$ *are monotone, nondecreasing functions defined for* $0 \leqslant t < \infty$. *Then, for* $\mathbf{u}_i(0) \in H_g$, *one has*

$$\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_g^2 \leqslant e^{\rho_1(t)} \left(\| \mathbf{u}_1(0) - \mathbf{u}_2(0) \|_g^2 + 2\| A_g^{-\frac{1}{2}}(\mathbf{f}_1 - \mathbf{f}_2) \|_\infty^2 \, t\right)$$

and for $\mathbf{u}_i(0) \in V_g$, *one has*

$$\| A_g^{\frac{1}{2}}(\mathbf{u}_1(t) - \mathbf{u}_2(t)) \|_g^2 \leqslant e^{\rho_2(t)}(\| A_g^{\frac{1}{2}}(\mathbf{u}_1(0) - \mathbf{u}_2(0)) \|_g^2 + 2\| \mathbf{f}_1 - \mathbf{f}_2 \|_\infty^2 t).$$

Here, we denote

$$\rho_1(t) = \int_0^t 2\left(d_1\| A_g^{\frac{1}{2}}\mathbf{u}_1(s) \|_g^2 + \frac{\| \nabla g \|_\infty^2}{m^2}\right) ds$$

and

$$\rho_2(t) = \int_0^t 2\left(\frac{\| \nabla g \|_\infty^2}{m^2} + \frac{d_2}{\lambda_1^2}\| A_g\mathbf{u}_1(s) \|_g^2 + \frac{d_3}{\lambda_1^2}\| A_g^{\frac{1}{2}}\mathbf{u}_2(s) \|_g^2\right) ds,$$

where λ_1 *is the first eigenvalue of* A_g.

Remark 1. Set

$$\alpha_1 = \alpha_1(g) = \lambda_1 - \frac{2}{m^2}\| \nabla g \|_\infty^2, \quad \alpha_2 = \alpha_2(g) = \frac{2}{\lambda_1 \alpha_1}, \tag{14}$$

where λ_1 is the first eigenvalue of A_g and

$$\| \nabla g \|_\infty = \sup_{(x,y)\in\Omega} |\nabla g(x,y)|.$$

Then, by Proposition 2, we have $\lambda_1 \geqslant \frac{4\pi^2 m}{M}$. So, if $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ then

$$\alpha_1 = \lambda_1 - \frac{2}{m^2}\| \nabla g \|_\infty^2 \geqslant \frac{4\pi^2 m}{M} - \frac{2}{m^2}\| \nabla g \|_\infty^2 > \frac{2\pi^2 m}{M}$$

and

$$\alpha_2 = \frac{2}{\lambda_1 \alpha_1} < \frac{M^2}{4\pi^4 m^2}.$$

Therefore, if $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ then we can choose α_1, α_2 only depend on m, M.

Then, for small enough $\| \nabla g \|_\infty$, i.e., g "close" to 1, in the following proposition we see the dissipativity of the solutions in the spaces H_g, V_g and $\mathcal{D}(A_g)$.

Proposition 6. *We assume that* $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ *and* $\mathbf{f} \in L^2(\Omega, g)$. *Then the following hold*:

1. *For* $\mathbf{u}_0 \in H_g$, *one has*

$$\| \mathbf{u}(t) \|_g^2 \leqslant e^{-\alpha_1 t} \| \mathbf{u}_0 \|_g^2 + \alpha_2 \| \mathbf{f} \|_g^2 \tag{15}$$

and

$$\left(1 - \frac{m}{2M}\right) \int_{t_1}^{t} \| A_g^{\frac{1}{2}} \mathbf{u}(s) \|_g^2 \, ds \leqslant \| \mathbf{u}(t_1) \|_g^2 + \frac{2(t - t_1)}{\lambda_1} \| \mathbf{f} \|_g^2$$

for $0 \leqslant t_1 \leqslant t < \infty$.

2. *For* $\mathbf{u}_0 \in V_g$ *then there exist constants,* $r_1 = r_1(m, M, \mathbf{f})$, $r_2 = r_2(m, M, \mathbf{f})$ *and* $L_1 = L_1(m, M, \mathbf{f})$($L_1$ *does not depend on* $\mathbf{u}_0$) *such that for* $t \geqslant 0$,

$$\| A_g^{\frac{1}{2}} \mathbf{u}(t) \|_g^2 \leqslant r_1 (1 + \| A_g^{\frac{1}{2}} \mathbf{u}_0 \|_g^2) e^{-\alpha_1 t} + L_1. \tag{16}$$

In addition, if $\mathbf{u}_0 \in \mathcal{D}(A_g)$ *and the forcing term* $\mathbf{f} \in V_g$ *then there exists constants* $r_3 = r_3(m, M, \mathbf{f})$ *and* $L_2 = L_2(m, M, \mathbf{f})$($L_2$ *does not depend on* $\mathbf{u}_0$) *such that*

$$\| A_g\mathbf{u}(t) \|_g^2 \leqslant r_3(1 + \| A_g\mathbf{u}_0 \|_g^2)e^{-\alpha_1 t} + L_2 \quad for\, t \geqslant 0. \tag{17}$$

As a result of the Proposition 6, we are in the position to prove the existence of the global attractors for the semiflows generated by the weak and strong solutions. Here, we assume that the forcing term $\mathbf{f}$ is a time-independent function. We will let $\sigma_w(t, \mathbf{u}_0) = S_w(t)\mathbf{u}_0$ denoted the semiflows on H_g generated by a weak solution on with the data $(\mathbf{u}_0, \mathbf{f})$ where $\mathbf{u}_0 \in H_g$ and $\mathbf{f} \in L^2(\Omega, g)$. Likewise, let $\sigma_s(t, \mathbf{u}_0) = S_s(t)\mathbf{u}_0$ denote the semiflows on V_g generated by a strong solution with the data $(\mathbf{u}_0, \mathbf{f})$, where $\mathbf{u}_0 \in V_g$ and $\mathbf{f} \in L^2(\Omega, g)$.

Proposition 7. *Let* $\mathbf{f} \in L^2(\Omega, g)$ *and we assume that* $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$. *Then, for* $\mathbf{u}_0 \in H_g$, $\sigma_w(t, \mathbf{u}_0) = S_w(t)\mathbf{u}_0$ *is a semiflow on* H_g *which is point dissipative and compact for* $t > 0$. *Also, there exists a global attractor* $\mathcal{A}_w$ *for* $S_w(t)$ *and the semiflow* $S_w(t)$ *is robust at* $\mathcal{A}_w$ *for every* $\mathbf{f} \in L^2(\Omega, g)$.

Likewise for $\mathbf{u}_0 \in V_g$, $\sigma_s(t, \mathbf{u}_0) = S_s(t)\mathbf{u}_0$ *is a semiflow on* V_g *which is point dissipative and compact for* $t > 0$. *Furthermore, there exists a global attractor* $\mathcal{A}_s$ *for* $S_s(t)$ *and the semiflow* $S_s(t)$ *is robust at* $\mathcal{A}_s$ *for every* $\mathbf{f} \in L^2(\Omega, g)$. *In addition, we note that* $\mathcal{A}_s = \mathcal{A}_w$, *for fixed* $\mathbf{f} \in L^2(\Omega, g)$.

4. Robustness of the global attractors

In this section, we will study the behavior of the global attractors of the g-Navier–Stokes equations as $g \to 1$ in the proper sense.

Before we do that we will first review useful definitions and propositions developed in Section 2.3 of Sell and You [8]. Then, we will describe two main theorems of this paper. In Section 4.1, we prove useful lemmas for main theorems. In Section 4.2, we will prove the first main theorem that the semiflows generated by the weak solutions of the g-Navier–Stokes equations is robust at the global attractor of the semiflows generated by the weak solutions of the Navier–Stokes equations. In Section 4.3, we will prove the results of Section 4.2 for the strong solutions.

Definition 1. Let Λ be a metric space. We will say that S_λ, for $\lambda \in \Lambda$, is a *continuous family of semiflows* on M, provided that $S_\lambda(t)u = \sigma(\lambda, u, t)$, and the mapping $\sigma : \Lambda \times M \times [0, \infty) \to M$ satisfies the following conditions:

1. the restriction mapping $\sigma : \Lambda \times M \times (0, \infty) \to M$ is continuous.
2. for each $\lambda \in \Lambda$, the mapping $S_\lambda(t)$ is a semiflow on M.

We will say that the semiflow $S_0(t)$ is *imbedded into a continuous family* of semiflows S_λ, for $\lambda \in \Lambda$, provided that there is a $\lambda_0 \in \Lambda$ such that

$$S_{\lambda_0}(t)u = S_0(t)u \quad \text{for } u \in M \text{ and } t \in [0, \infty).$$

Definition 2. Let $\mathcal{A}_0$ be an attractor for a given semiflow $S_0(t)$ on the Banach space W. Let $S_0(t)$ be imbedded into a continuous family of semiflows $S_\lambda(t)$, where $\lambda \in \Lambda$ and $S_{\lambda_0}(t) = S_0(t)$. We will say that the family $S_\lambda(t)$ is *robust* at $\mathcal{A}_0$, with respect to λ at $\lambda = \lambda_0$, provided that, for every $\varepsilon > 0$, there is a neighborhood $O = O(\varepsilon)$ of λ_0 in Λ such that for each $\lambda \in O$, the semiflow $S_\lambda(t)$ has an attractor $\mathcal{A}_\lambda$ and

$$\mathcal{A}_\lambda \subset N_\varepsilon(\mathcal{A}_0) \quad \text{for all } \lambda \in O,$$

where

$$N_\varepsilon(\mathcal{A}_0) = \{\mathbf{u} \in W \ : \ dist_W(\mathbf{u}, \mathcal{A}_0) \leqslant \varepsilon\}.$$

Definition 3. Let σ be a semiflow on $M \subset W$. We will say that σ is *asymptotically compact* on a set $B \subset M$, if for any sequences $\mathbf{u}_n \in B$ and $t_n \to \infty$, there exist subsequences, which we relabel as $\mathbf{u}_n$ and t_n, with the property that the limit $\mathbf{v} = \lim S(t_n)\mathbf{u}_n$ exist and $\mathbf{v} \in M$.

Proposition 8. *Let σ be a semiflow on $M \subset W$. Let $\mathcal{A}$ be a nonempty, compact set in M, and assume that $\mathcal{A}$ attract a nonempty set B uniformly. Then σ is asymptotically compact on B.*

Proposition 9 (*Robustness theorem*). *Let $S_0(t)$ be a semiflow on the Banach space W and let $\mathcal{A}_0$ be an attractor for $S_0(t)$. We also let U_1 be any fixed, bounded neighborhood of $\mathcal{A}_0$ and let $S_0(t)$ be imbedded into any continuous family $S_\lambda(t)$, where each semiflow $S_\lambda(t)$, for $\lambda \in \Lambda$, is asymptotically compact on U_1. Then the family $S_\lambda(t)$ is robust at $\mathcal{A}_0$.*

Let us go back to our problem. We define $\tilde{\sigma}_w(g, \mathbf{v}, t)$ on H_1 by

$$\tilde{\sigma}_w(g, \mathbf{v}, t) = P_1\sigma_w(g, \ \ P_g\mathbf{v}, \ \ t),$$

where $\sigma_w(g, P_g\mathbf{v}, t)$ is a semiflow on the space H_g generated by the weak solutions of Eq. (10) with the initial condition $P_g\mathbf{v}$. We will see later that $\tilde{\sigma}_w(g, \mathbf{v}, t)$ is a semiflow on H_1. Then, we prove that the family of the semiflows $\tilde{\sigma}_w(g, \mathbf{v}, t)$ with respect to g is robust at the global attractor of the semiflow $\tilde{\sigma}_w(1, \mathbf{v}, t) = \sigma_w(1, \mathbf{v}, t)$.

Theorem I. *Let $\mathbf{f} \in L^2(\Omega)$ and $g \in \Lambda \subset W^{2,\infty}(\Omega)$, where Λ is given in Definition 4. Then, for every $g \in \Lambda$, $\tilde{\sigma}_w(g, \mathbf{v}, t)$ has a global attractor and the family of the semiflows with respect to g, $\tilde{\sigma}_w(g, \mathbf{v}, t)$, is robust at the global attractor of the semiflow $\tilde{\sigma}_w(1, \mathbf{v}, t)$.*

Also, we can define the semiflow on V_1 by

$$\tilde{\sigma}_s(g, \mathbf{v}, t) = P_1\sigma_s(g, \ \ P_g\mathbf{v}, \ \ t),$$

where $\sigma_s(g, P_g\mathbf{v}, t)$ is a semiflow on the space V_g generated by the strong solutions of Eq. (10) with the initial condition $P_g\mathbf{v}$. And we get the following theorem, due to Robustness theorem.

Theorem II. *Let $\mathbf{f} \in L^2(\Omega)$ and $g \in \Lambda \subset W^{2,\infty}(\Omega)$, where Λ is given in Definition 4. Then, for every $g \in \Lambda$, $\tilde{\sigma}_s(g, \mathbf{v}, t)$ has a global attractor and the family of the semiflows with respect to g, $\tilde{\sigma}_s(g, \mathbf{v}, t)$, is robust at the global attractor of the semiflow $\tilde{\sigma}_s(1, \mathbf{v}, t)$.*

4.1. Useful lemmas

In this section, one should recall that we denote by H_1, V_1, P_1, A_1 instead of H_g, V_g, P_g, A_g for the constant function $g = 1$.

By using the fact that the pressure space Q does not depend on the function g we can have the following.

Lemma 1. *Assume that $\nabla p \in Q$ and $p \in H^3(\Omega)$. Then we have*

$$P_g\left[\frac{d}{dt}(\nabla p(t))\right] = \frac{d}{dt}P_g(\nabla p(t)) = 0,$$
$$P_g[-\Delta(\nabla p(t))] = P_g[\nabla(-\Delta p(t))] = 0,$$
$$P_g[(\nabla p(t) \cdot \nabla)\nabla p(t)] = P_g\left[\nabla\left(\frac{1}{2}(\nabla p(t) \cdot \nabla p(t))\right)\right] = 0.$$

One should note that Lemma 1 also holds for the constant function $g = 1$.

Lemma 2. *For every $\mathbf{u}_1, \mathbf{u}_2 \in H_g$, if $P_1\mathbf{u}_1 = P_1\mathbf{u}_2$ then $\mathbf{u}_1 = \mathbf{u}_2$. Also, for $\mathbf{v}_1, \mathbf{v}_2 \in H_1$, if $P_g\mathbf{v}_1 = P_g\mathbf{v}_2$ then $\mathbf{v}_1 = \mathbf{v}_2$. In other words, $P_1P_g(\mathbf{v}) = \mathbf{v}$, for $\mathbf{v} \in H_1$, and $P_gP_1(\mathbf{u}) = \mathbf{u}$, for $\mathbf{u} \in H_g$.*

Proof. Assume that $P_1\mathbf{u}_1 = P_1\mathbf{u}_2$. Then, by (8) there exist $\nabla p_i \in Q$, for $i = 1, 2$, such that $\mathbf{u}_i = P_1\mathbf{u}_i + \nabla p_i$, because $\mathbf{k} = \int_\Omega \mathbf{u}_i \, d\mathbf{x} = 0$. So, we have $\mathbf{u}_1 - \mathbf{u}_2 = \nabla(p_1 - p_2)$. But $\mathbf{u}_1 - \mathbf{u}_2 \in H_g$ and $\nabla(p_1 - p_2) \in H_g^\perp$. Therefore, $\mathbf{u}_1 - \mathbf{u}_2 = 0$. Similarly by using (7) one can prove the others. $\square$

For the periodic boundary condition we consider the pressure term $p(\mathbf{x})$ with $\int_\Omega p(\mathbf{x}) \, d\mathbf{x} = 0$. Therefore, for the smooth enough function $p(\mathbf{x})$ one obtains

$$4\pi^2 \parallel \nabla(p) \parallel^2 \leqslant \parallel \Delta p \parallel^2 \quad \text{and} \quad c_1 \parallel p \parallel_{H^2} \leqslant \parallel \Delta p \parallel \leqslant c_2 \parallel p \parallel_{H^2} \tag{18}$$

for some constants c_i, $i = 1, 2$. Then, as a corollary of the previous lemma we can obtain the following.

Corollary 1. *Assume that* $\mathbf{u} \in H_g$ *with*

$$\mathbf{u} = \mathbf{v} + \nabla p \quad for\ \mathbf{v} \in H_1, \quad \nabla p \in Q. \tag{19}$$

Then there exist constants $c_3 = c_3(m, M)$ *and* $c_4 = c_4(m, M)$ *such that*

$$\| \Delta p \| \leqslant c_3 \| \nabla g \|_\infty \| \mathbf{u} \|, \quad \| p \|_{H^2(\Omega)} \leqslant c_4 \| \nabla g \|_\infty \| \mathbf{u} \|. \tag{20}$$

In addition, we have $c_5 = c_5(m, M)$ *and* $c_6 = c_6(m, M)$ *such that*

$$\| \Delta p \| \leqslant c_5 \| \nabla g \|_\infty \| \mathbf{v} \|, \quad \| p \|_{H^2(\Omega)} \leqslant c_6 \| \nabla g \|_\infty \| \mathbf{v} \|. \tag{21}$$

Proof. By taking the divergence $\nabla \cdot$ to the both sides of (19), one obtains

$$\frac{\nabla g}{g} \cdot \mathbf{u} = -\Delta p \tag{22}$$

and one can easily get (20).

Next, by using Lemma 2 we have that $P_g \mathbf{v} = \mathbf{u}$ which lead to

$$\| \mathbf{u} \| \leqslant \frac{1}{\sqrt{m}} \| \mathbf{u} \|_g \leqslant \frac{1}{\sqrt{m}} \| \mathbf{v} \|_g \leqslant \frac{\sqrt{M}}{\sqrt{m}} \| \mathbf{v} \|$$

so that one can obtains (21). □

Remark 2. By (22), for $\mathbf{u} \in H^\alpha(\Omega)$, one has a constant $\delta_0 = \delta_0(m, M)$ such that $\| p \|_{H^{\alpha+2}} \leqslant \delta_0 \| g \|_{W^{\alpha+1,\infty}} \| \mathbf{u} \|_{H^\alpha}$.

Lemma 3. *We assume that* $\int_\Omega \frac{1}{g}\, d\mathbf{x} = 1$. *Then, for* $\mathbf{u} \in L^2(\Omega)$ *we have*

$$P_1 P_g \mathbf{u} = P_1 \mathbf{u} - P_1 \left(\frac{\mathbf{k}}{g} \right), \tag{23}$$

where $\mathbf{k} = \int_\Omega \mathbf{u}\, d\mathbf{x}$. *As a result,* $P_1 P_g \mathbf{u} = P_1 \mathbf{u}$ *if* $\int_\Omega \mathbf{u}\, d\mathbf{x} = 0$.

Furthermore, for $\mathbf{u} \in L^2(\Omega)$ *and* $\mathbf{w} \in H_1$ *we have*

$$|\langle P_1 P_g \mathbf{u}, \mathbf{w} \rangle| \leqslant |\langle \mathbf{u}, \mathbf{w} \rangle| + \frac{\| 1 - g \|_\infty}{m} \| \mathbf{k} \| \, \| \mathbf{w} \|. \tag{24}$$

Proof. By (7), for every $\mathbf{u} \in L^2(\Omega) = L^2(\Omega, g)$ there exist $\mathbf{v} \in H_g$ and $\nabla q \in Q$ such that

$$\mathbf{u} = \mathbf{v} + \frac{\mathbf{k}}{g} + \nabla q, \quad \text{where } \mathbf{k} = \int_\Omega \mathbf{u}\, d\mathbf{x},$$

which implies $P_g\mathbf{u} = \mathbf{v}$. Since $P_1(\nabla q) = 0$ and the projection P_1 is linear operator we get (23).

Since $\langle \mathbf{k}, \mathbf{w}\rangle = 0$ for $\mathbf{w} \in H_1$ and the projection P_1 is symmetric we have

$$\begin{aligned} |\langle P_1 P_g \mathbf{u}, \mathbf{w}\rangle| &= |\langle P_1\mathbf{u}, \mathbf{w}\rangle| + \left|\left\langle P_1\left(\frac{\mathbf{k}}{g}\right), \mathbf{w}\right\rangle\right| = |\langle \mathbf{u}, \mathbf{w}\rangle| + \left|\left\langle \frac{\mathbf{k}}{g}, \mathbf{w}\right\rangle\right| \\ &= |\langle \mathbf{u}, \mathbf{w}\rangle| + \left|\left\langle \frac{\mathbf{k}}{g} - \mathbf{k}, \mathbf{w}\right\rangle\right| = |\langle \mathbf{u}, \mathbf{w}\rangle| + \left|\int_\Omega (\frac{1}{g} - 1)\mathbf{k}\cdot\mathbf{w}\, d\mathbf{x}\right| \\ &\leqslant |\langle \mathbf{u}, \mathbf{w}\rangle| + \left|\left|\frac{1}{g} - 1\right|\right|_\infty \|\mathbf{k}\| \; \|\mathbf{w}\| \end{aligned}$$

which implies (24). □

Remark 3. From (23) we can have the following instead of (24),

$$|\langle P_1 P_g \mathbf{u}, \mathbf{w}\rangle| \leqslant |\langle \mathbf{u}, \mathbf{w}\rangle| + \frac{1}{m}\|\mathbf{k}\| \; \|\mathbf{w}\|. \tag{25}$$

In fact, in this paper we need and will use inequality (25) to simplify the calculations.

In next lemma, we will see the relationship between two spaces H_{g_i} for different g_i, $i = 1, 2$.

Lemma 4. *We assume that* $|\nabla g|_\infty^2 < \frac{m^3\pi^2}{M}$. *We also let* $\mathbf{u}_i \in H_{g_i}$ *with*

$$\mathbf{u}_i = \mathbf{v}_i \; + \; \nabla p_i \quad for \mathbf{v}_i \in H_1, \quad \nabla p_i \in Q \tag{26}$$

for $i = 1, 2$ *and* $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. *Then we have positive constant* η *such that*

$$\eta \; \| p_1 - p_2 \|_{H^2} \leqslant \left|\left| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right|\right|_\infty \|\mathbf{u}_1\| + \left|\left|\frac{\nabla g_2}{g_2}\right|\right|_\infty \|\mathbf{w}\|, \tag{27}$$

where $\eta = c_1 \left(1 - \frac{\|\nabla g_2\|_\infty}{2\pi m}\right)$ *and* c_1 *is given in* (18).

Proof. Since $\mathbf{v}_i \in H_1$, by taking ∇ both sides in Eq. (26) one obtains

$$\frac{\nabla g_1}{g_1} \cdot \mathbf{u}_1 - \frac{\nabla g_2}{g_2} \cdot \mathbf{u}_2 = -\Delta(p_1 - p_2). \tag{28}$$

Then, by using the fact

$$\| \mathbf{u}_1 - \mathbf{u}_2 \| \leqslant \| \mathbf{v}_1 - \mathbf{v}_2 \| + \| \nabla(p_1 - p_2) \| = \| \mathbf{w} \| + \| \nabla(p_1 - p_2) \|,$$

one has from (18) and (28) that

$$\begin{aligned}
\| \Delta(p_1 - p_2) \| &= \left\| \frac{\nabla g_1}{g_1} \cdot \mathbf{u}_1 - \frac{\nabla g_2}{g_2} \cdot \mathbf{u}_2 \right\| \\
&\leqslant \left\| \frac{\nabla g_1}{g_1} \cdot \mathbf{u}_1 - \frac{\nabla g_2}{g_2} \cdot \mathbf{u}_1 \right\| + \left\| \frac{\nabla g_2}{g_2} \cdot \mathbf{u}_1 - \frac{\nabla g_2}{g_2} \cdot \mathbf{u}_2 \right\| \\
&\leqslant \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 \| + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 - \mathbf{u}_2 \| \\
&\leqslant \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 \| + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty (\| \mathbf{w} \| + \| \nabla(p_1 - p_2) \|) \\
&\leqslant \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 \| + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty (\| \mathbf{w} \| + \frac{1}{2\pi} \| \Delta(p_1 - p_2) \|),
\end{aligned}$$

which implies

$$\left(1 - \frac{\| \nabla g_2 \|_\infty}{2\pi m}\right) \| \Delta(p_1 - p_2) \| \leqslant \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 \| + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{w} \|.$$

Thus, by (18) one obtains (27). □

Next, we want to see the relationship between the norms in the spaces H_g and H_1 as well as in the spaces V_g and V_1. Before we do next lemma we recall that $\| A_g^{\frac{1}{2}} \mathbf{u} \|_g = \| \nabla \mathbf{u} \|_g$ and for the case of the constant function $g = 1$, one has that for $\mathbf{v} \in V_1$,

$$2\pi \| \mathbf{v} \| \leqslant \| \nabla \mathbf{v} \| = \| A_1^{\frac{1}{2}} \mathbf{v} \|$$

and for $\mathbf{v} \in \mathcal{D}(A_1)$,

$$A_1 \mathbf{v} = P_1(-\Delta \mathbf{v}) = -\Delta \mathbf{v}.$$

Lemma 5. *We let* $\mathbf{u} \in H_g$ *with*

$$\mathbf{u} = \mathbf{v} + \nabla p \quad \textit{for} \quad \mathbf{v} \in H_1, \quad \nabla p \in Q. \tag{29}$$

Then the followings hold:
1. *One has*

$$\frac{1}{M} \; \| \mathbf{u} \|_g^2 \leqslant \| \mathbf{v} \|^2 \leqslant \frac{1}{m} \; \| \mathbf{u} \|_g^2. \tag{30}$$

2. *For* $\mathbf{u} \in V_g$, *we have*

$$\mathbf{u} = \mathbf{v} \; + \; \nabla p, \quad \mathbf{v} \in V_1, \quad \nabla p \in Q \tag{31}$$

and

$$\| \nabla \mathbf{u} \|^2 = \| \nabla \mathbf{v} \|^2 + \| \nabla (\nabla p) \|^2. \tag{32}$$

In addition, if $|\nabla g|_\infty^2 < \frac{m^3\pi^2}{M}$ *then one has*

$$l_1 \; \| A_g^{\frac{1}{2}} \mathbf{u} \|_g^2 \leqslant \| A_1^{\frac{1}{2}} \mathbf{v} \|^2 \leqslant \frac{1}{m} \; \| A_g^{\frac{1}{2}} \mathbf{u} \|_g^2, \tag{33}$$

where

$$l_1 = l_1(g) = \frac{4\pi^2}{M \; (4\pi^2 + c_6^2 \| \nabla g \|_\infty^2)}.$$

3. *For* $\mathbf{u} \in \mathcal{D}(A_g)$, *we have*

$$\mathbf{u} = \mathbf{v} + \nabla p, \quad \mathbf{v} \in \mathcal{D}(A_1), \quad \nabla p \in Q. \tag{34}$$

In addition, if $|\nabla g|_\infty^2 < \frac{m^3\pi^2}{M}$ *then one has*

$$l_2 \; \| A_g \mathbf{u} \|_g^2 \leqslant \| A_1 \mathbf{v} \|^2 \leqslant l_3 \; \| A_g \mathbf{u} \|_g^2, \tag{35}$$

where

$$l_2 = l_2(g) = \frac{4\pi^4 m^2}{M \left(2\pi^2 m + 2\pi \| \nabla g \|_\infty + c_6 \| \nabla g \|_\infty^2\right)^2}$$

and

$$l_3 = l_3(g) = \frac{(m\sqrt{\lambda_1^g} + 2\| \nabla g \|_\infty)^2}{m^3 \lambda_1^g},$$

λ_1^g is the smallest eigenvalue of A_g.

Proof. One can easily get (30). To get (31), we take $\nabla\cdot$ into (29) and we have $-\Delta p = \frac{\nabla g}{g} \cdot \mathbf{u}$. So, for given $\mathbf{u} \in V_g = H^1(\Omega, g) \cap H_g$, one obtains $p \in H^3(\Omega)$ which implies $\mathbf{v} \in V_1(\Omega) = H^1(\Omega) \cap H_1$. Also, by integration by parts, we have

$$\langle \nabla \mathbf{v}, \nabla(\nabla p)\rangle = \int_\Omega \nabla \mathbf{v} \cdot \nabla(\nabla p)\, d\mathbf{x} = \int_\Omega \mathbf{v} \cdot \nabla(\Delta p)\, d\mathbf{x} = 0.$$

Thus, by (31), one obtains Eq. (32),

$$\| \nabla \mathbf{u} \|^2 = \langle \nabla \mathbf{v}, \nabla \mathbf{v}\rangle + \langle \nabla(\nabla p), \nabla(\nabla p)\rangle = \| \nabla \mathbf{v} \|^2 + \| \nabla(\nabla p) \|^2,$$

which implies

$$\| A_1^{\frac{1}{2}} \mathbf{v} \|^2 = \| \nabla \mathbf{v} \|^2 \leqslant \| \nabla \mathbf{u} \|^2 \leqslant \frac{1}{m} \| \nabla \mathbf{u} \|_g^2 = \frac{1}{m} \| A_g^{\frac{1}{2}} \mathbf{u} \|_g^2.$$

Moreover, one has from (21) that

$$\begin{aligned} \| \nabla \mathbf{u} \|^2 &= \| \nabla \mathbf{v} \|^2 + \| \nabla(\nabla p) \|^2 \leqslant \| \nabla \mathbf{v} \|^2 + \| p \|_{H^2(\Omega)}^2 \\ &\leqslant \| \nabla \mathbf{v} \|^2 + c_6^2 \| \nabla g \|_\infty^2 \| \mathbf{v} \|^2 \leqslant \left(1 + \frac{c_6^2 \| \nabla g \|_\infty^2}{4\pi^2}\right) \| \nabla \mathbf{v} \|^2, \end{aligned}$$

which complete the proof of (33). Similar to (31) one can obtain (34). To prove (35) one take $-\Delta$ to the both sides of (34) to get

$$-\Delta \mathbf{v} = -\Delta \mathbf{u} + \Delta(\nabla p).$$

Then, since $-\Delta \mathbf{v} \in H_1$, one has from Lemma 2 that $P_1 P_g(-\Delta \mathbf{v}) = -\Delta \mathbf{v}$. So, by Lemma 1, one has

$$\begin{aligned} \| A_1 \mathbf{v} \| &= \| -\Delta \mathbf{v} \| = \| P_1 P_g(-\Delta \mathbf{v}) \| \leqslant \| P_g(-\Delta \mathbf{v}) \| = \| P_g(-\Delta \mathbf{u} + \Delta(\nabla p)) \| \\ &= \| P_g(-\Delta \mathbf{u}) \| = \left\| P_g \left(-\Delta_g \mathbf{u} + \left(\frac{\nabla g}{g} \cdot \nabla\right) \mathbf{u}\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\sqrt{m}}\left(\| P_g(-\Delta_g \mathbf{u}) \|_g + \left\| P_g\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)\right)\right\|_g\right)\\
&\leqslant \frac{1}{\sqrt{m}}\left(\| A_g \mathbf{u} \|_g + \left\|\left(\frac{\nabla g}{g}\cdot\nabla\right)\mathbf{u}\right\|_g\right)\\
&\leqslant \frac{1}{\sqrt{m}}\left(\| A_g \mathbf{u} \|_g + \frac{2\| \nabla g \|_\infty}{m}\| \nabla \mathbf{u} \|_g\right)\\
&\leqslant \frac{1}{\sqrt{m}}\left(\| A_g \mathbf{u} \|_g + \frac{2\| \nabla g \|_\infty}{m\sqrt{\lambda_1^g}}\| A_g \mathbf{u} \|_g\right)\\
&= \frac{1}{\sqrt{m}}\left(1 + \frac{2|\nabla g|_\infty}{m\sqrt{\lambda_1^g}}\right)\| A_g \mathbf{u} \|_g,
\end{aligned}
$$

which implies the right-hand side of (35). To prove the left inequality of (35) we take $-\Delta_g$ to the both sides of (34) to get

$$
-\Delta_g \mathbf{u} = -\Delta_g \mathbf{v} - \Delta_g(\nabla p). \tag{36}
$$

One should recall $-\Delta_g \mathbf{v} = -\Delta \mathbf{v} - \left(\frac{\nabla g}{g}\cdot\nabla\right)\mathbf{v}$. So, one has from the fact $A_1\mathbf{v} = -\Delta\mathbf{v}$ that

$$
\begin{aligned}
\| -\Delta_g \mathbf{v} \| &\leqslant \| -\Delta \mathbf{v} \| + \left\|\left(\frac{\nabla g}{g}\cdot\nabla\right)\mathbf{v}\right\| \leqslant \| -\Delta \mathbf{v} \| + 2\,\frac{\| \nabla g \|_\infty}{m}\,\| \nabla \mathbf{v} \|\\
&\leqslant \left(1 + \frac{\| \nabla g \|_\infty}{\pi m}\right)\| -\Delta \mathbf{v} \| = \left(1 + \frac{\| \nabla g \|_\infty}{\pi m}\right)\| A_1 \mathbf{v} \|,
\end{aligned}
$$

which implies

$$
\| -\Delta_g \mathbf{v} \| \leqslant \left(1 + \frac{\| \nabla g \|_\infty}{\pi m}\right)\| A_1 \mathbf{v} \|. \tag{37}
$$

Then, one has from Lemma 1 that

$$
P_g(-\Delta_g(\nabla p)) = P_g[-\Delta(\nabla p)] + P_g\left[\left(\frac{\nabla g}{g}\cdot\nabla\right)\nabla p\right] = P_g\left[\left(\frac{\nabla g}{g}\cdot\nabla\right)\nabla p\right]. \tag{38}
$$

Hence, one has from (38) that

$$\| P_g(-\Delta_g(\nabla p)) \|_g \leqslant \left\| P_g \left[\left(\frac{\nabla g}{g} \cdot \nabla \right) \nabla p \right] \right\|_g \leqslant \left\| \left(\frac{\nabla g}{g} \cdot \nabla \right) \nabla p \right\|_g$$

$$\leqslant \sqrt{M} \left\| \left(\frac{\nabla g}{g} \cdot \nabla \right) \nabla p \right\| \leqslant \frac{2\sqrt{M} \| \nabla g \|_\infty}{m} \| p \|_{H^2(\Omega)}.$$

Then, by (21), one obtains

$$\| P_g(-\Delta_g(\nabla p)) \|_g \leqslant \frac{2\sqrt{M} \| \nabla g \|_\infty}{m} \| p \|_{H^2} \leqslant \frac{2c_6 \sqrt{M} \| \nabla g \|_\infty^2}{m} \| \mathbf{v} \|,$$

which implies

$$\| P_g(-\Delta_g(\nabla p)) \|_g \leqslant \frac{c_6\sqrt{M} \| \nabla g \|_\infty^2}{2\pi^2 m} \| A_1 \mathbf{v} \|. \tag{39}$$

Therefore, by (36), (37) and (39), one has

$$\begin{aligned} \| A_g \mathbf{u} \|_g &= \| P_g(-\Delta_g \mathbf{u}) \|_g \leqslant \| P_g(-\Delta_g \mathbf{v}) \|_g + \| P_g(-\Delta_g(\nabla p)) \|_g \\ &\leqslant \| -\Delta_g \mathbf{v} \|_g + \| P_g(-\Delta_g(\nabla p)) \|_g \\ &\leqslant \sqrt{M} \| -\Delta_g \mathbf{v} \| + \| P_g(-\Delta_g(\nabla p)) \|_g \\ &\leqslant \sqrt{M} \left(1 + \frac{\| \nabla g \|_\infty}{\pi m} \right) \| A_1 \mathbf{v} \| + \frac{c_6\sqrt{M} \| \nabla g \|_\infty^2}{2\pi^2 m} \| A_1 \mathbf{v} \|, \end{aligned}$$

which complete the proof of the left inequality of (35). □

Remark 4. Since $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ one can obtain l_1, l_2, and l_3 which only depend on m and M but not $\| \nabla g \|_\infty$.

4.2. Proof of Theorem I

First, we define new set Λ as the following:

Definition 4. Let us define the set Λ with the metric inherited from $W^{1,\infty}(\Omega)$ as $g \in \Lambda$ if

1. $g(\mathbf{x}) \in C_{\text{per}}^\infty(\Omega)$ and $\int_\Omega \frac{1}{g}\, d\mathbf{x} = 1$ with $0 < m \leqslant g(x, y) \leqslant M$, for all $(x, y) \in \Omega$.
2. $\| g \|_{W^{1,\infty}}^2 < \frac{m^3\pi^2}{M}$ and $\| g \|_{W^{2,\infty}} \leqslant M_0$ for some constant M_0.

Note that in Definition 4, the constant function $g = 1$ belong to the set Λ and the condition $\int_\Omega \frac{1}{g}\, d\mathbf{x} = 1$ is to simplify the calculations.

We define $\tilde{\sigma}_w(g, \mathbf{v}, t)$ on H_1 by

$$\tilde{\sigma}_w(g, \mathbf{v}, t) = P_1 \sigma_w(g, \;\; P_g \mathbf{v}, \;\; t),$$

where $\sigma_w(g, P_g\mathbf{v}, t)$ is a semiflow on the space H_g generated by weak solutions of the Eq. (10) with the initial condition $P_g\mathbf{u}$. Then, $\tilde{\sigma}_w(g, \mathbf{v}, t)$ is a semiflow on H_1.

Lemma 6. *Let* $\mathbf{f} \in L^2(\Omega)$ *be a time-independent function. Then* $\tilde{\sigma}_w(g, \mathbf{v}, t)$ *is a semiflow on* H_1*, for any fixed* $g \in \Lambda$*.*

Proof. Since $\sigma_w(g, P_g\mathbf{v}, t)$ is a semiflow on H_g, one has by Lemma 2 that

$$\tilde{\sigma}_w(g, \mathbf{v}, 0) = P_1(\sigma_w(g, P_g\mathbf{v}, 0)) = P_1(P_g\mathbf{v}) = \mathbf{v}$$

for all $\mathbf{v} \in H_1$ and that for $s, t \geqslant 0$,

$$\begin{aligned}
\tilde{\sigma}_w(g, \tilde{\sigma}_w(g, \mathbf{v}, s), t) &= P_1[\sigma_w(g, P_g P_1(\sigma_w(g, P_g\mathbf{v}, s)), t)] \\
&= P_1[\sigma_w(g, \sigma_w(g, P_g\mathbf{v}, s), t)] = P_1[\sigma_w(g, P_g\mathbf{v}, s+t)] = \tilde{\sigma}_w(g, \mathbf{v}, s+t).
\end{aligned}$$

Next, we have from (5) that

$$\begin{aligned}
\| \tilde{\sigma}_w(g, \mathbf{v}_1, t) - \tilde{\sigma}_w(g, \mathbf{v}_2, t) \|^2 &= \| P_1\sigma_w(g, P_g\mathbf{v}_1, t) - P_1\sigma_w(g, P_g\mathbf{v}_2, t) \|^2 \\
&= \| P_1[\sigma_w(g, P_g\mathbf{v}_1, t) - \sigma_w(g, P_g\mathbf{v}_2, t)] \|^2 \leqslant \| \sigma_w(g, P_g\mathbf{v}_1, t) - \sigma_w(g, P_g\mathbf{v}_2, t) \|^2 \\
&\leqslant \frac{1}{m} \| \sigma_w(g, P_g\mathbf{v}_1, t) - \sigma_w(g, P_g\mathbf{v}_2, t) \|_g^2 \leqslant \frac{1}{m} e^{\rho_1(t)} \| P_g\mathbf{v}_1 - P_g\mathbf{v}_2 \|_g^2 \\
&\leqslant \frac{1}{m} e^{\rho_1(t)} \| \mathbf{v}_1 - \mathbf{v}_2 \|_g^2 \leqslant \frac{M}{m} e^{\rho_1(t)} \| \mathbf{v}_1 - \mathbf{v}_2 \|^2,
\end{aligned}$$

which implies the continuity with respect to $\mathbf{v}$, for fixed $t > 0$. Also, one has

$$\begin{aligned}
\| \tilde{\sigma}_w(g, \mathbf{v}, t_1) - \tilde{\sigma}_w(g, \mathbf{v}, t_2) \|^2 &= \| P_1\sigma_w(g, P_g\mathbf{v}, t_1) - P_1\sigma_w(g, P_g\mathbf{v}, t_2) \|^2 \\
&\leqslant \| \sigma_w(g, P_g\mathbf{v}, t_1) - \sigma_w(g, P_g\mathbf{v}, t_2) \|^2 \leqslant \frac{1}{m} \; \| \sigma_w(g, P_g\mathbf{v}, t_1) - \sigma_w(g, P_g\mathbf{v}, t_2) \|_g^2.
\end{aligned}$$

Now, since $\sigma_w(g, P_g\mathbf{v}, t)$ is a semiflow on H_g, $\sigma_w(g, P_g\mathbf{v}, t)$ is continuous with respect to t. Therefore, $\tilde{\sigma}_w(g, P_g\mathbf{v}, t)$ is continuous with respect to t. $\square$

Lemma 7. *Assume that $g \in \Lambda$ and the forcing term $\mathbf{f} \in L^2(\Omega)$ be a time independent function. Let $\mathbf{u}(t)$ be a weak solution of the Eq. (10) with the initial condition $P_g\mathbf{u}_0$, where $\mathbf{u}_0 \in H_1$. Then, for any $t_0 > 0$, there exist $\delta_1 = \delta_1(\mathbf{u}_0, t_0, m, M)$, $\delta_2 = \delta_2(\mathbf{u}_0, t_0, m, M)$, $\delta_3 = \delta_3(\mathbf{u}_0, t_0, m, M)$ which do not depend on $\| \nabla g \|_\infty$, such that*
1. $\| P_1\mathbf{u}(t) \|$, $\| \mathbf{u}(t) \|_g \leqslant \delta_1$ *for all* $0 < t_0 \leqslant t < \infty$.
2. *If* $\mathbf{u}_0 \in V_1$ *then* $\| A_1^{\frac{1}{2}} P_1\mathbf{u}(t) \|$, $\| A_g^{\frac{1}{2}}\mathbf{u}(t) \|_g \leqslant \delta_2$ *for all* $0 < t_0 \leqslant t < \infty$.
3. *If* $\mathbf{u}_0 \in \mathcal{D}(A_1)$ *then* $\| A_1 P_1\mathbf{u}(t) \|$, $\| A_g\mathbf{u}(t) \|_g \leqslant \delta_3$ *for all* $0 < t_0 \leqslant t < \infty$.

Proof. One can obtain δ_1 by (15) and (30), δ_2 by (16) and (33). One can also get δ_3 by (17) and (35). $\square$

Now, we want to show that the semiflow $\tilde{\sigma}_w(g, P_g\mathbf{v}, t)$ is continuous with respect to g.

Lemma 8. *Let the forcing term $\mathbf{f} \in L^2(\Omega)$ be a time-independent function with $\int_\Omega \mathbf{f}\, d\mathbf{x} = 0$. Then the semiflows*

$$\tilde{\sigma}_w(g, \mathbf{v}, t) : \Lambda \times H_1 \times (0, \infty) \to H_1$$

is continuous.

Proof. Let $\mathbf{v}_0 \in H_1$ and $g_i \in \Lambda$, for $i = 1, 2$. Also, we denote by $\mathbf{u}_i \in H_{g_i}$ for the weak solution of Eq. (10) with the initial condition $P_{g_i}\mathbf{v}_0$. Then, for the solution $\mathbf{u}_i \in H_{g_i}$, we can rewrite by

$$\mathbf{u}_i(t) = \mathbf{v}_i(t) + \nabla p_i(t), \quad \nabla p_i(t) \in Q, \quad \mathbf{v}_i(t) \in H_1.$$

Since $\mathbf{u}_i(t)$ is a strong solution of Eq. (10) for $t \geqslant t_0 > 0$, one has from (34) $\mathbf{u}_i(t) \in \mathcal{D}(A_g)$, $\mathbf{v}_i(t) \in \mathcal{D}(A_1)$ and $p_i(t) \in H^3(\Omega)$ for $t \geqslant t_0 > 0$. Also, since

$$\int_\Omega (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\, d\mathbf{x} = \int_\Omega (\mathbf{v}_i \cdot \nabla)\nabla p_i\, d\mathbf{x} = \int_\Omega \mathbf{f}\, d\mathbf{x} = 0,$$

we have

$$P_1 P_g(\mathbf{v}_i \cdot \nabla)\mathbf{v}_i = P_1(\mathbf{v}_i \cdot \nabla)\mathbf{v}_i, \quad P_1 P_g(\mathbf{v}_i \cdot \nabla)\nabla p_i = P_1(\mathbf{v}_i \cdot \nabla)\nabla p_i, \quad P_1 P_g\mathbf{f} = P_1\mathbf{f}.$$

Thus, by the Eq. (1), Lemmas 1 and 3 we have

$$\frac{d\mathbf{v}_i}{dt} + A_1\mathbf{v}_i + P_1(\mathbf{v}_i \cdot \nabla)\mathbf{v}_i + P_1 P_g(\nabla p_i \cdot \nabla)\mathbf{v}_i + P_1(\mathbf{v}_i \cdot \nabla)\nabla p_i = P_1\mathbf{f} \tag{40}$$

for $i = 1, 2$ and $t \geqslant t_0 > 0$.

We denote by $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$ and get

$$\begin{aligned}&\frac{d\mathbf{w}}{dt} + A_1\mathbf{w} + P_1(\mathbf{v}_1\cdot\nabla)\mathbf{w} + P_1(\mathbf{w}\cdot\nabla)\mathbf{v}_2\\ &\quad + P_1(\mathbf{w}\cdot\nabla)\nabla p_1 + P_1(\mathbf{v}_2\cdot\nabla)\nabla(p_1-p_2)\\ &\quad + P_1P_g(\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1 + P_1P_g(\nabla p_2\cdot\nabla)\mathbf{w} = 0 \qquad (41)\end{aligned}$$

for $t \geqslant t_0 > 0$. Then, by taking the scalar product with $\mathbf{w}$ to both sides of (41) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\| \mathbf{w} \|^2 + \| A_1^{\frac{1}{2}}\mathbf{w} \|^2\\ &\leqslant |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}_2,\mathbf{w}\rangle| + |\langle(\mathbf{w}\cdot\nabla)\nabla p_1,\mathbf{w}\rangle| + |\langle(\mathbf{v}_2\cdot\nabla)\nabla(p_1-p_2),\mathbf{w}\rangle|\\ &\quad + |\langle P_1P_g(\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1,\mathbf{w}\rangle| + |\langle P_1P_g(\nabla p_2\cdot\nabla)\mathbf{w},\mathbf{w}\rangle|\\ &= |I| + |II| + |III| + |IV| + |V| \qquad (42)\end{aligned}$$

for $t \geqslant t_0 > 0$. Now by Lemma 7, (12), the Sobolev imbedding inequality and the Young inequality we obtain

$$\begin{aligned}|I| &= |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}_2,\mathbf{w}\rangle| \leqslant 2 \| \mathbf{w} \| \, \| \nabla\mathbf{w} \| \, \| \nabla\mathbf{v}_2 \|\\ &\leqslant \tfrac{1}{8}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + 16\| A_1^{\frac{1}{2}}\mathbf{v}_2 \|^2 \| \mathbf{w} \|^2\\ &\leqslant \tfrac{1}{8}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + c_7\| \mathbf{w} \|^2 \qquad (43)\end{aligned}$$

for some positive constants $c_7 = c_7(t_0, \mathbf{v}_0, m, M)$. Since $g \in \Lambda$, by (12), (20), Lemma 7, the Sobolev imbedding inequality and the Young inequality we have

$$\begin{aligned}|II| &= |\langle(\mathbf{w}\cdot\nabla)\nabla p_1,\mathbf{w}\rangle| \leqslant 2 \| \mathbf{w} \| \, \| \nabla\mathbf{w} \| \, \| p_1 \|_{H^2}\\ &\leqslant \tfrac{1}{8}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + 16c_4^2\| \nabla g \|_\infty^2 \| \mathbf{u}_g \|^2 \| \mathbf{w} \|^2\\ &\leqslant \tfrac{1}{8}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + c_8\| \nabla g \|_\infty^2 \, \| \mathbf{w} \|^2\\ &\leqslant \tfrac{1}{8}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \tilde{c}_8 \, \| \mathbf{w} \|^2 \qquad (44)\end{aligned}$$

for some positive constants $\tilde{c}_8 = \tilde{c}_8(t_0, \mathbf{v}_0, m, M)$ and $c_8 = c_8(t_0, \mathbf{v}_0, m, M)$.

Now, by (25) we obtain

$$|V| = |\langle P_1P_g(\nabla p_2\cdot\nabla)\mathbf{w},\mathbf{w}\rangle| = |\langle(\nabla p_2\cdot\nabla)\mathbf{w},\mathbf{w}\rangle| + \frac{1}{m}\| \mathbf{k} \| \, \| \mathbf{w} \|, \qquad (45)$$

where $\mathbf{k} = \int_\Omega (\nabla p_2\cdot\nabla)\mathbf{w}\, d\mathbf{x}$.

Then, by the Sobolev imbedding inequality, (20) and Lemma 7, we have

$$\begin{aligned}|\langle(\nabla p_2\cdot\nabla)\mathbf{w},\mathbf{w}\rangle| &\leqslant \|\nabla p_2\|_{L^4}\ \|\nabla\mathbf{w}\|\|\mathbf{w}\|_{L^4}\leqslant\|p_2\|_{H^2}\ \|\mathbf{w}\|^{\frac{1}{2}}\|\nabla\mathbf{w}\|^{\frac{3}{2}}\\ &\leqslant 4c_4\|\nabla g\|_\infty\ \|\mathbf{u}_g\|\ \|\mathbf{w}\|^{\frac{1}{2}}\|\nabla\mathbf{w}\|^{\frac{3}{2}}\\ &\leqslant \tfrac{1}{8}\|A_1^{\frac{1}{2}}\mathbf{w}\|^2+c_9\|\nabla g\|_\infty^4\|\mathbf{w}\|^2 \end{aligned}\tag{46}$$

for some constant $c_9=c_9(t_0,\mathbf{v}_0,m,M)$.

Since, by using the integration by parts, (20) and Lemma 7, we can get

$$\begin{aligned}\frac{1}{m}\|\mathbf{k}\| &\leqslant \frac{1}{m}\left|\int_\Omega(\nabla p_2\cdot\nabla)\mathbf{w}\,d\mathbf{x}\right|\leqslant\frac{1}{m}\left|\int_\Omega(\Delta p_2)\mathbf{w}\,d\mathbf{x}\right|\\ &\leqslant \frac{1}{m}\|\Delta p_2\|\ \|\mathbf{w}\|\leqslant\frac{c_4}{m}\|\nabla g\|_\infty\|\mathbf{u}_g\|\ \|\mathbf{w}\|\\ &\leqslant c_{10}\|\nabla g\|_\infty\|\mathbf{w}\| \end{aligned}\tag{47}$$

for some constant $c_{10}=c_{10}(t_0,\mathbf{v}_0,m,M)$.

Hence, by (45)–(47) we obtain

$$\begin{aligned}|V| &\leqslant \tfrac{1}{8}\|A_1^{\frac{1}{2}}\mathbf{w}\|^2+(c_9\|\nabla g\|_\infty^4+c_{10}\|\nabla g\|_\infty)\|\mathbf{w}\|^2\\ &\leqslant \tfrac{1}{8}\|A_1^{\frac{1}{2}}\mathbf{w}\|^2+\tilde{c}_{10}\|\mathbf{w}\|^2\end{aligned}\tag{48}$$

for some constant $\tilde{c}_{10}=\tilde{c}_{10}(t_0,\mathbf{v}_0,m,M)$, because $g\in\Lambda$.

Next, Since $g\in\Lambda$ one can get by Lemma 7, (12), (27) and the Young inequality that

$$\begin{aligned}|III| &= |\langle(\mathbf{v}_2\cdot\nabla)\nabla(p_1-p_2),\mathbf{w}\rangle|\\ &\leqslant 4\|\mathbf{v}_2\|^{\frac{1}{2}}\|A_1^{\frac{1}{2}}\mathbf{v}_2\|^{\frac{1}{2}}\|p_1-p_2\|_{H^2}\|\mathbf{w}\|^{\frac{1}{2}}\|A_1^{\frac{1}{2}}\mathbf{w}\|^{\frac{1}{2}}\\ &\leqslant c_{11}\|p_1-p_2\|_{H^2}\|\mathbf{w}\|^{\frac{1}{2}}\|A_1^{\frac{1}{2}}\mathbf{w}\|^{\frac{1}{2}}\leqslant c_{11}\|p_1-p_2\|_{H^2}\|A_1^{\frac{1}{2}}\mathbf{w}\|\\ &\leqslant \frac{c_{11}}{\eta}\left(\left\|\frac{\nabla g_1}{g_1}-\frac{\nabla g_2}{g_2}\right\|_\infty\|\mathbf{u}_1\|+\left\|\frac{\nabla g_2}{g_2}\right\|_\infty\|\mathbf{w}\|\right)\|A_1^{\frac{1}{2}}\mathbf{w}\|\\ &\leqslant \frac{1}{4}\|A_1^{\frac{1}{2}}\mathbf{w}\|^2+c_{12}\left\|\frac{\nabla g_1}{g_1}-\frac{\nabla g_2}{g_2}\right\|_\infty^2+c_{13}\left\|\frac{\nabla g_2}{g_2}\right\|_\infty^2\|\mathbf{w}\|^2\\ &\leqslant \frac{1}{4}\|A_1^{\frac{1}{2}}\mathbf{w}\|^2+c_{12}\left\|\frac{\nabla g_1}{g_1}-\frac{\nabla g_2}{g_2}\right\|_\infty^2+c_{14}\|\mathbf{w}\|^2\end{aligned}\tag{49}$$

for some constants $c_i=c_i(t_0,\mathbf{v}_0,m,M)$ $i=11,12,13,14$.

By (25) we have

$$|IV| = |\langle P_1 P_g(\nabla(p_1 - p_2)\cdot\nabla)\mathbf{v}_1, \mathbf{w}\rangle| \leqslant |\langle(\nabla(p_1 - p_2)\cdot\nabla)\mathbf{v}_1, \mathbf{w}\rangle| + \frac{1}{m}\|\,\mathbf{k}\,\|\;\|\,\mathbf{w}\,\|, \tag{50}$$

where $\mathbf{k} = \int_\Omega(\nabla(p_1 - p_2)\cdot\nabla)\mathbf{v}_1\,d\mathbf{x}$. Similar to $|III|$ we can obtain

$$\begin{aligned} &|\langle(\nabla(p_1 - p_2)\cdot\nabla)\mathbf{v}_1, \mathbf{w}\rangle| \\ &\leqslant 4\|\,p_1 - p_2\,\|_{H^2}\|\,A_1^{\frac12}\mathbf{v}_1\,\|\|\,A_1^{\frac12}\mathbf{w}\,\| \\ &\leqslant \frac14\|\,A_1^{\frac12}\mathbf{w}\,\|^2 + c_{15}\left\|\frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right\|_\infty^2 + c_{16}\|\,\mathbf{w}\,\|^2 \end{aligned} \tag{51}$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M)$, $i = 15, 16$, because $g \in \Lambda$.

Also, we have from the integration by parts, (27) and Lemma 7 that

$$\begin{aligned} \|\,\mathbf{k}\,\| &= \left|\int_\Omega(\nabla(p_1 - p_2)\cdot\nabla)\mathbf{v}_1\,d\mathbf{x}\right| = \left|\int_\Omega(\Delta(p_1 - p_2))\mathbf{v}_1\,d\mathbf{x}\right| \\ &\leqslant \|\,\Delta(p_1 - p_2)\,\|\;\|\,\mathbf{v}_1\,\| \leqslant \|\,p_1 - p_2\,\|_{H^2}\|\,\mathbf{v}_1\,\| \\ &\leqslant \frac{\delta_1}{\eta}\left(\left\|\frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right\|_\infty \delta_1 + \left\|\frac{\nabla g_2}{g_2}\right\|_\infty \|\,\mathbf{w}\,\|\right). \end{aligned} \tag{52}$$

So, we can obtain

$$\begin{aligned} \frac1m\|\,\mathbf{k}\,\|\;\|\,\mathbf{w}\,\| &\leqslant \frac{\delta_1}{m\eta}\left(\left\|\frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right\|_\infty \delta_1 + \left\|\frac{\nabla g_2}{g_2}\right\|_\infty \|\,\mathbf{w}\,\|\right)\|\,\mathbf{w}\,\| \\ &\leqslant \tilde{c}_{15}\left\|\frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right\|_\infty^2 + \tilde{c}_{16}\|\,\mathbf{w}\,\|^2 \end{aligned} \tag{53}$$

for some constants $\tilde{c}_i = \tilde{c}_i(t_0, \mathbf{v}_0, m, M)$, $i = 15, 16$, because $g \in \Lambda$.

So by (50)–(53) we obtains

$$|IV| \leqslant \frac14\|\,A_1^{\frac12}\mathbf{w}\,\|^2 + c_{17}\left\|\frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2}\right\|_\infty^2 + c_{18}\|\,\mathbf{w}\,\|^2 \tag{54}$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M)$, $i = 17, 18$.

Thus, for $t \geqslant t_0 > 0$, by (42)–(44), (48), (49) and (54), one has

$$\frac{1}{2}\frac{d}{dt}\| \mathbf{w} \|^2 + \left(\frac{\pi^2}{2} - c_7 - \tilde{c}_8 - \tilde{c}_{10} - c_{14} - c_{18}\right) \| \mathbf{w} \|^2$$
$$\leqslant (c_{12} + c_{17}) \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2,$$

which implies that

$$\frac{d}{dt}\| \mathbf{w} \|^2 + \beta_1 \ \| \mathbf{w} \|^2 \leqslant \beta_2 \quad \text{for } t \geqslant t_0 > 0,$$

where

$$\beta_1 = \pi^2 - 2c_7 - 2\tilde{c}_8 - 2\tilde{c}_{10} - 2c_{14} - 2c_{18},$$
$$\beta_2 = 2(c_{12} + c_{17}) \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2.$$

So, by the Gronwall inequality, one has

$$\| \mathbf{w}(t) \|^2 \leqslant \| \mathbf{w}(0) \|^2 e^{-\beta_1 t} + C(t) \ \beta_2 \quad \text{for } t \geqslant t_0 > 0,$$

where $C(t)$ is a positive bounded t-function. But, we have $\mathbf{w}(0) = \mathbf{v}_1(0) - \mathbf{v}_2(0) = \mathbf{v}_0 - \mathbf{v}_0 = 0$ and by the definition of β_2, $\beta_2 \to 0$ as $\| g_1 - g_2 \|_{W^{1,\infty}} \to 0$. Hence, for any fixed $t \geqslant t_0 > 0$ and $\mathbf{v}_0 \in H_1$, $\| \mathbf{w}(t) \|^2$ goes to zero as $\| g_1 - g_2 \|_{W^{1,\infty}} \to 0$. It means that for the fixed $\mathbf{v}_0 \in H_1$ and $t \geqslant t_0 > 0$, we have

$$\begin{aligned} \| \tilde{\sigma}_w(g_1, \mathbf{v}_0, t) - \tilde{\sigma}_w(g_2, \mathbf{v}_0, t) \|^2 &= \| P_1\sigma_w(g_1, P_{g_1}\mathbf{v}_0, t) - P_1\sigma_w(g_2, P_{g_2}\mathbf{v}_0, t) \|^2 \\ &= \| P_1\mathbf{u}_1(t) - P_1\mathbf{u}_2(t) \|^2 = \| \mathbf{v}_1(t) - \mathbf{v}_2(t) \|^2 \\ &= \| \mathbf{w}(t) \|^2 \leqslant C(t)\beta_2, \end{aligned}$$

which goes to zero as $\| g_1 - g_2 \|_{W^{1,\infty}} \to 0$. Hence, the solution $\tilde{\sigma}_w(g, \mathbf{v}, t)$ on the space H_1 is continuous in terms of $g \in \Lambda$.

Therefore, by Lemma 6, for any $\varepsilon > 0$ there exist δ such that

$$\| \tilde{\sigma}_w(g_1, \mathbf{v}_1, t_1) - \tilde{\sigma}_w(g_2, \mathbf{v}_2, t_2) \|^2$$
$$\leqslant \| \tilde{\sigma}_w(g_1, \mathbf{v}_1, t_1) - \tilde{\sigma}_w(g_2, \mathbf{v}_1, t_1) \|^2 + \| \tilde{\sigma}_w(g_2, \mathbf{v}_1, t_1) - \tilde{\sigma}_w(g_2, \mathbf{v}_2, t_2) \|^2 \leqslant \varepsilon,$$

whenever

$$\begin{aligned}
&\| (g_1, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_2, t_2) \|_{\Lambda \times H_1 \times (0,\infty)} \\
&\quad \leqslant \| (g_1, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_1, t_1) \|_{\Lambda \times H_1 \times (0,\infty)} \\
&\qquad + \| (g_2, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_2, t_2) \|_{\Lambda \times H_1 \times (0,\infty)} \leqslant \delta.
\end{aligned}$$

Thus, we complete the proof. □

Now, let us go back to Remark 1 and Proposition 6.

Let $\mathbf{u}_0 \in H_1$ and $\mathbf{u}_g = \mathbf{v}_g + \nabla p_g$ be a weak solution of the Eq. (10) with the initial data $\mathbf{u}_g(0) = P_g \mathbf{u}_0$. Then, for $g \in \Lambda$, by (15), there exists some $t_g = t_g(\mathbf{u}_0) \geqslant 0$ such that

$$\| \mathbf{u}_g(t) \|_g^2 \leqslant 2\alpha_2 \| \mathbf{f} \|_g^2 \leqslant \frac{M^2}{2\pi^4 m^2} \| \mathbf{f} \|_g^2 \quad \text{for} \quad t \geqslant t_g(\mathbf{u}_0).$$

Also, by (30) one obtains

$$\| \mathbf{v}_g \|^2 = \| P_1 \mathbf{u}_g \|^2 \leqslant \frac{M^3}{2\pi^4 m^3} \| \mathbf{f} \|^2 \quad \text{for} \quad t \geqslant t_g(\mathbf{u}_0).$$

Definition 5. We define the set $U_w \subset H_1$ as

$$U_w = \left\{ \mathbf{v} \in H_1 \ : \ \| \mathbf{v} \|^2 \leqslant \frac{M^3}{2\pi^4 m^3} \| \mathbf{f} \|^2 \right\}.$$

Proof of Theorem I. First, for $g \in \Lambda$, one can see easily from the definition of $\tilde{\sigma}_w(g, \mathbf{v}, t)$ that the semiflow $\tilde{\sigma}_w(g, \mathbf{v}, t)$ on H_1 has a global attractor. In fact, $P_1 \mathcal{A}_g$ is the global attractor of the semiflow $\tilde{\sigma}_w(g, \mathbf{v}, t)$, where $\mathcal{A}_g$ is the global attractor of the semiflow generated by weak solutions of (10).

Also, by Lemmas 6 and 8, one note that $\tilde{\sigma}_w(1, \mathbf{v}, t) = \sigma_w(1, \mathbf{v}, t)$ be imbedded into continuous family $\tilde{\sigma}_w(g, \mathbf{v}, t)$.

Then, by the definition of the set U_w, for $g \in \Lambda$, $\tilde{\sigma}_w(g, \mathbf{v}, t)$ is asymptotically compact on U_w because the global attractor $P_1 \mathcal{A}_g$ of the semiflow $\tilde{\sigma}_w(g, \mathbf{v}, t)$ is contained in the set U_w.

Therefore, by the Robustness theorem, Proposition 9 that $\tilde{\sigma}_w(g, \mathbf{v}, t)$ is robust at $P_1 \mathcal{A}_1 = \mathcal{A}_1$ the attractor of $\tilde{\sigma}_w(1, \mathbf{v}, t) = \sigma_w(1, \mathbf{v}, t)$. One should note that $\mathcal{A}_1$ is the global attractor of the 2D Navier–Stokes equations. □

4.3. Proof of Theorem II

We define $\tilde{\sigma}_s(g, \mathbf{v}, t)$ on V_1 by

$$\tilde{\sigma}_s(g, \mathbf{v}, t) = P_1\sigma_s(g, \; P_g\mathbf{v}, \; t), \tag{55}$$

where $\sigma_s(g, P_g\mathbf{v}, t)$ is a semiflow on the space V_g generated by the strong solutions of the Eq. (10) with the initial condition $P_g\mathbf{u}$.

Lemma 9. *Let the forcing term* $\mathbf{f} \in L^2(\Omega)$ *be a time-independent function. Then* $\tilde{\sigma}_s(g, \mathbf{v}, t)$ *is a semiflow on* V_1, *for any fixed* $g \in \Lambda$.

Proof. Since $\sigma_s(g, P_g\mathbf{v}, t)$ is a semiflow on V_g, one has by Lemma 2 that

$$\tilde{\sigma}_s(g, \mathbf{v}, 0) = P_1(\sigma_s(g, P_g\mathbf{v}, 0)) = P_1(P_g\mathbf{v}) = \mathbf{v}$$

for all $\mathbf{v} \in V_1$. Similar to Lemma 6, one can obtains that

$$\tilde{\sigma}_s(g, \tilde{\sigma}_s(g, \mathbf{v}, s), t) = \tilde{\sigma}_s(g, \mathbf{v}, s + t), \quad s, t \geqslant 0.$$

Next, to prove the continuity of $\tilde{\sigma}_s$ with respect to $\mathbf{u}$, we have from (5) and (33) that

$$\begin{aligned}
&\| A_1^{\frac{1}{2}}(\tilde{\sigma}_s(g, \mathbf{v}_1, t) - \tilde{\sigma}_s(g, \mathbf{v}_2, t)) \|^2 \\
&= \| A_1^{\frac{1}{2}}(P_1\sigma_s(g, P_g\mathbf{v}_1, t) - P_1\sigma_s(g, P_g\mathbf{v}_2, t)) \|^2 \\
&= \| A_1^{\frac{1}{2}}(P_1[\sigma_s(g, P_g\mathbf{v}_1, t) - \sigma_s(g, P_g\mathbf{v}_2, t)]) \|^2 \\
&\leqslant \frac{1}{m} \| A_g^{\frac{1}{2}}[\sigma_s(g, P_g\mathbf{v}_1, t) - \sigma_s(g, P_g\mathbf{v}_2, t)] \|_g^2 \\
&\leqslant \frac{1}{m} e^{\rho_2(t)} \| A_g^{\frac{1}{2}}(P_g\mathbf{v}_1 - P_g\mathbf{v}_2) \|_g^2 \leqslant \frac{1}{ml_1} e^{\rho_2(t)} \| A_1^{\frac{1}{2}}(\mathbf{v}_1 - \mathbf{v}_2) \|^2,
\end{aligned}$$

which implies the continuity with respect to $\mathbf{u}$ on the space V_1, for fixed $t > 0$. Also, by (33), one obtains

$$\begin{aligned}
&\| A_1^{\frac{1}{2}}(\tilde{\sigma}_s(g, \mathbf{v}, t_1) - \tilde{\sigma}_s(g, \mathbf{v}, t_2)) \|^2 \\
&= \| A_1^{\frac{1}{2}}(P_1\sigma_s(g, P_g\mathbf{v}, t_1) - P_1\sigma_s(g, P_g\mathbf{v}, t_2)) \|^2 \\
&\leqslant \frac{1}{m} \| A_g^{\frac{1}{2}}(\sigma_s(g, P_g\mathbf{v}, t_1) - \sigma_s(g, P_g\mathbf{v}, t_2)) \|_g^2.
\end{aligned}$$

Since $\sigma_s(g, P_g\mathbf{v}, t)$ is a semiflow on V_g, $\sigma_s(g, P_g\mathbf{v}, t)$ is continuous with respect to t and hence $\tilde{\sigma}_s(g, \mathbf{v}, t)$ is continuous with respect to t, on the space V_1. $\square$

Now, we want to show that the semiflow $\tilde{\sigma}_s(g, \mathbf{v}, t)$ is continuous with respect to g, on the space V_1.

Lemma 10. *Let the forcing term* $\mathbf{f} \in L^2(\Omega)$ *be a time-independent function with* $\int_\Omega \mathbf{f}\, d\mathbf{x} = 0$. *Then the semiflows*

$$\tilde{\sigma}_s(g, \mathbf{v}, t) : \Lambda \times V_1 \times (0, \infty) \to V_1$$

is continuous.

Proof. By taking the scalar product of $A_1\mathbf{w}$ into the Eq. (41), one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \| A_1\mathbf{w} \|^2 &\leqslant |\langle(\mathbf{v}_1 \cdot \nabla)\mathbf{w}, A_1\mathbf{w}\rangle| + |\langle(\mathbf{w} \cdot \nabla)\mathbf{v}_2, A_1\mathbf{w}\rangle| \\ &\quad + |\langle(\mathbf{w} \cdot \nabla)\nabla p_1, A_1\mathbf{w}\rangle| + |\langle(\mathbf{v}_2 \cdot \nabla)\nabla(p_1 - p_2), A_1\mathbf{w}\rangle| \\ &\quad + |\langle P_1 P_g(\nabla(p_1 - p_2) \cdot \nabla)\mathbf{v}_1, A_1\mathbf{w}\rangle| + |\langle P_1 P_g(\nabla p_2 \cdot \nabla)\mathbf{w}, A_1\mathbf{w}\rangle| \\ &= |I| + |II| + |III| + |IV| + |V| + |VI| \end{aligned} \tag{56}$$

for $t \geqslant t_0 > 0$. Now, by applying the Young inequality, Proposition 3 and Lemma 7, we have

$$\begin{aligned}|I| &= |\langle(\mathbf{v}_1 \cdot \nabla)\mathbf{w}, A_1\mathbf{w}\rangle| \leqslant \gamma_2 \| A_1\mathbf{v}_1 \| \, \| A_1^{\frac{1}{2}}\mathbf{w} \| \, \| A_1\mathbf{w} \| \\ &\leqslant \tfrac{1}{8}\| A_1\mathbf{w} \|^2 + c_{19}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \end{aligned} \tag{57}$$

for some constant $c_{19} = c_{19}(t_0, \mathbf{v}_0, m, M)$. Similar to $|I|$ we can obtains by the Young inequality, Proposition 3 and Lemma 7 that

$$\begin{aligned}|II| &= |\langle(\mathbf{w} \cdot \nabla)\mathbf{v}_2, A_1\mathbf{w}\rangle| \leqslant \gamma_2 \| A_1^{\frac{1}{2}}\mathbf{w} \| \, \| A_1\mathbf{v}_2 \| \, \| A_1\mathbf{w} \| \\ &\leqslant \tfrac{1}{8}\| A_1\mathbf{w} \|^2 + c_{20}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \end{aligned} \tag{58}$$

for some constant $c_{20} = c_{20}(t_0, \mathbf{v}_0, m, M)$.

Before we estimate $|III|$, since $g \in \Lambda$ one note from (22) and Remark 2 that

$$\| p_i \|_{H^3} \leqslant \delta_0 \| g \|_{W^{2,\infty}} \| \mathbf{u}_i \|_{H^1} \leqslant \delta_0 M_0 \| \mathbf{u}_i \|_{H^1} \quad \text{for } i = 1, 2 \tag{59}$$

for some positive constant $\delta_0 = \delta_0(m, M)$. Now, by the Young inequality, Proposition 3, Lemma 7, (13) and (59) we have

$$\begin{aligned} |III| &= |\langle(\mathbf{w}\cdot\nabla)\nabla p_1, A_1\mathbf{w}\rangle| \leqslant \gamma_1 \| \mathbf{w} \|_{H^1} \, \| A_1\mathbf{w} \| \, \| p_1 \|_{H^3} \\ &\leqslant \gamma_1\delta_0 M_0 \| \mathbf{w} \|_{H^1} \, \| A_1\mathbf{w} \| \, \| \mathbf{u}_1 \|_{H^1} \\ &\leqslant \gamma_1\delta_0\tilde{\delta}^2 M_0 \| A_g^{\frac{1}{2}}\mathbf{u}_1 \|_g \| A_1^{\frac{1}{2}}\mathbf{w} \| \, \| A_1\mathbf{w} \| \\ &\leqslant c_{21} M_0 \| A_1^{\frac{1}{2}}\mathbf{w} \| \, \| A_1\mathbf{w} \| \\ &\leqslant \tfrac{1}{8}\| A_1\mathbf{w} \|^2 + c_{22} \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \end{aligned} \tag{60}$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M, M_0)$, $i = 21, 22$ because $g \in \Lambda$.

Next, by (25) we have

$$\begin{aligned} |VI| &= |\langle P_1 P_g(\nabla p_2\cdot\nabla)\mathbf{w}, A_1\mathbf{w}\rangle| \\ &\leqslant |\langle(\nabla p_2\cdot\nabla)\mathbf{w}, A_1\mathbf{w}\rangle| + \frac{1}{m}\| \mathbf{k} \| \, \| A_1\mathbf{w} \|, \end{aligned} \tag{61}$$

where $\mathbf{k} = \int_\Omega (\nabla p_2\cdot\nabla)\mathbf{w}\, d\mathbf{x}$.

Similar to $|III|$, by Proposition 3, Lemma 7, (13) and (59), there exist some constant $c_{23} = c_{23}(t_0, \mathbf{v}_0, m, M, M_0)$ such that

$$\begin{aligned} |\langle(\nabla p_2\cdot\nabla)\mathbf{w}, A_1\mathbf{w}\rangle| &\leqslant \gamma_1 \| p_2 \|_{H^3} \| \mathbf{w} \|_{H^1} \| A_1\mathbf{w} \| \\ &\leqslant \tfrac{1}{8}\| A_1\mathbf{w} \|^2 + c_{23} \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 . \end{aligned} \tag{62}$$

Then by (47) and the Young inequality we have

$$\begin{aligned} \frac{1}{m}\| \mathbf{k} \| \, \| A_1\mathbf{w} \| &\leqslant c_{10} \| \nabla g \|_\infty \| \mathbf{w} \| \, \| A_1\mathbf{w} \| \\ &\leqslant c_{10} \| \nabla g \|_\infty \| A_1^{\frac{1}{2}}\mathbf{w} \| \, \| A_1\mathbf{w} \| \\ &\leqslant \tfrac{1}{8}\| A_1\mathbf{w} \|^2 + c_{24} \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \end{aligned} \tag{63}$$

for some constant $c_{24} = c_{24}(t_0, \mathbf{v}_0, m, M)$ because $g \in \Lambda$.

So by (61)–(63) we have

$$|VI| \leqslant \tfrac{1}{4}\| A_1\mathbf{w} \|^2 + (c_{23} + c_{24}) \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 . \tag{64}$$

Next, to estimate $|IV|$, by Proposition 3, Lemma 7, (13) and (27), one can get

$$
\begin{aligned}
|IV| &= |\langle(\mathbf{v}_2\cdot\nabla)\nabla(p_1-p_2), A_1\mathbf{w}\rangle| \leqslant \gamma_1 \| \mathbf{v}_2 \|_{H^2} \| p_1-p_2 \|_{H^2} \| A_1\mathbf{w} \| \\
&\leqslant \tilde{\delta}\gamma_1 \| A_1\mathbf{v}_2 \| \, \| p_1-p_2 \|_{H^2} \| A_1\mathbf{w} \| \leqslant \gamma_1\tilde{\delta}\delta_3 \| A_1\mathbf{w} \| \, \| p_1-p_2 \|_{H^2} \\
&\leqslant \frac{\gamma_1\tilde{\delta}\delta_3}{\eta} \| A_1\mathbf{w} \| \left(\left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{u}_1 \| + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{w} \| \right) \\
&\leqslant \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{26} \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2 + c_{27} \left\| A_1^{\frac{1}{2}}\mathbf{w} \right\|^2
\end{aligned} \tag{65}
$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M)$, $i = 26, 27$ because $g \in \Lambda$.

To estimate $|V|$ we have by (25)

$$
\begin{aligned}
|V| &= |\langle P_1 P_g (\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1, A_1\mathbf{w}\rangle| \\
&\leqslant |\langle(\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1, A_1\mathbf{w}\rangle| + \frac{1}{m} \| \mathbf{k} \| \, \| A_1\mathbf{w} \|,
\end{aligned} \tag{66}
$$

where $\mathbf{k} = \int_\Omega (\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1 \, d\mathbf{x}$.

Then, similar to $|IV|$, by Proposition 3, Lemma 7, (13), (25) and (27) that

$$
\begin{aligned}
|\langle(\nabla(p_1-p_2)\cdot\nabla)\mathbf{v}_1, A_1\mathbf{w}\rangle| &\leqslant \gamma_1 \| p_1-p_2 \|_{H^2} \| \mathbf{v}_1 \|_{H^2} \| A_1\mathbf{w} \| \\
&\leqslant \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{28} \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2 + c_{29} \left\| A_1^{\frac{1}{2}}\mathbf{w} \right\|^2
\end{aligned} \tag{67}
$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M)$, $i = 28, 29$.

Also, by (47), (48) and the Young inequality we obtain

$$
\begin{aligned}
&\frac{1}{m} \| \mathbf{k} \| \, \| A_1\mathbf{w} \| \\
&\quad \leqslant \frac{\delta_1}{m\eta} \left(\left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty \delta_1 + \left\| \frac{\nabla g_2}{g_2} \right\|_\infty \| \mathbf{w} \| \right) \| A_1\mathbf{w} \| \\
&\quad \leqslant \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{30} \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2 + c_{31} \left\| A_1^{\frac{1}{2}}\mathbf{w} \right\|^2
\end{aligned} \tag{68}
$$

for some constants $c_i = c_i(t_0, \mathbf{v}_0, m, M)$, $i = 30, 31$.

Therefore, by (66)–(68) we get

$$
|V| \leqslant \frac{1}{4} \| A_1\mathbf{w} \|^2 + (c_{28}+c_{30}) \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2 + (c_{29}+c_{31}) \left\| A_1^{\frac{1}{2}}\mathbf{w} \right\|^2. \tag{69}
$$

Thus, for $t \geqslant t_0 > 0$, we have by (56)–(58), (60), (64), (65) and (69) that

$$\frac{d}{dt}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \leqslant 2(c_{19}+c_{20}+c_{22}+c_{23}+c_{24}+c_{27}+c_{29}+c_{31})\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + 2(c_{26}+c_{28}+c_{30}) \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2,$$

which implies that

$$\frac{d}{dt}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \leqslant \beta_3 \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \beta_4 \quad t \geqslant t_0 > 0,$$

where

$$\beta_3 = 2(c_{19}+c_{20}+c_{22}+c_{24}+c_{25}+c_{27}+c_{29}+c_{31}),$$
$$\beta_4 = 2(c_{26}+c_{28}+c_{30}) \left\| \frac{\nabla g_1}{g_1} - \frac{\nabla g_2}{g_2} \right\|_\infty^2.$$

Hence, by Gronwall inequality, one has

$$\| A_1^{\frac{1}{2}}\mathbf{w}(t) \|^2 \leqslant \| A_1^{\frac{1}{2}}\mathbf{w}(0) \|^2 e^{\beta_3 t} + \tilde{C}(t)\ \beta_4 \quad \text{for } t \geqslant t_0 > 0,$$

where $\tilde{C}(t)$ is some positive bounded t-function. But, we have $\mathbf{w}(0) = \mathbf{v}_1(0) - \mathbf{v}_2(0) = \mathbf{v}_0 - \mathbf{v}_0 = 0$ which implies $\| A_1^{\frac{1}{2}}\mathbf{w}(0) \| = 0$. Also, we note that by the definition of β_4, $\beta_4 \to 0$ as $\| g_1 - g_2 \|_{W^{1,\infty}} \to 0$. Therefore, $\| A_1^{\frac{1}{2}}\mathbf{w}(t) \|^2$ goes to zero as $\| g_1 - g_2 \|_{W^{1,\infty}} \to 0$. It means that for any fixed $\mathbf{v}_0 \in V_1$ and $t \geqslant t_0 > 0$, we have

$$\begin{aligned}
&\| A_1^{\frac{1}{2}}(\tilde{\sigma}_s(g_1, \mathbf{v}_0, t) - \tilde{\sigma}_s(g_2, \mathbf{v}_0, t)) \|^2 \\
&= \| A_1^{\frac{1}{2}}[P_1\sigma_s(g_1, P_{g_1}\mathbf{v}_0, t) - P_1\sigma_s(g_2, P_{g_2}\mathbf{v}_0, t)] \|^2 \\
&= \| A_1^{\frac{1}{2}}(P_1\mathbf{u}_1(t) - P_1\mathbf{u}_2(t)) \|^2 = \| A_1^{\frac{1}{2}}(\mathbf{v}_1(t) - \mathbf{v}_2(t)) \|^2 \\
&= \| A_1^{\frac{1}{2}}\mathbf{w}(t) \|^2 \leqslant \tilde{C}(t)\beta_4,
\end{aligned}$$

which goes to zero as $\| g_1 - g_2 \|_{W^{1,\infty}}$ goes to zero. Hence, the solution $\tilde{\sigma}_s(g, \mathbf{v}, t)$ on the space V_1 is continuous in terms of $g \in \Lambda$.

Therefore, by Lemma 9, for any $\varepsilon > 0$ there exist δ such that

$$\begin{aligned}&\| A_1^{\frac{1}{2}}(\tilde{\sigma}_s(g_1, \mathbf{v}_1, t_1) - \tilde{\sigma}_s(g_2, \mathbf{v}_2, t_2)) \|^2 \\ &\quad\leqslant \| A_1^{\frac{1}{2}}(\tilde{\sigma}_w(g_1, \mathbf{v}_1, t_1) - \tilde{\sigma}_w(g_2, \mathbf{v}_1, t_1)) \|^2 \\ &\qquad + \| A_1^{\frac{1}{2}}(\tilde{\sigma}_w(g_2, \mathbf{v}_1, t_1) - \tilde{\sigma}_w(g_2, \mathbf{v}_2, t_2)) \|^2 \leqslant \varepsilon,\end{aligned}$$

whenever

$$\begin{aligned}&\| (g_1, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_2, t_2) \|_{\Lambda \times V_1 \times (0,\infty)} \\ &\quad\leqslant \| (g_1, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_1, t_1) \|_{\Lambda \times V_1 \times (0,\infty)} \\ &\qquad + \| (g_2, \mathbf{v}_1, t_1) - (g_2, \mathbf{v}_2, t_2) \|_{\Lambda \times V_1 \times (0,\infty)} \leqslant \delta.\end{aligned}$$

Thus, we complete the proof. □

Let $\mathbf{v}_0 \in V_1$ and $\mathbf{u}_g$ is a strong solution of the Eq. (10) with the initial data $\mathbf{u}_g(0) = P_g \mathbf{v}_0$. Then by (16), there exist $0 \leqslant t_g = t_g(\mathbf{v}_0)$ and L_1 such that

$$\| A_g^{\frac{1}{2}} \mathbf{u}_g(t) \|_g^2 \leqslant 2L_1 \quad \text{for all } t \geqslant t_g,$$

which implies by (33) that

$$\| A_1^{\frac{1}{2}} P_1 \mathbf{u}_g \|^2 \leqslant \frac{2L_1}{m} \quad \text{for all } t \geqslant t_g,$$

where $g \in \Lambda$. One should note that in the inequality (16) we can choose the constant L_1 which depends only on m and M, but not $\| \nabla g \|_\infty$ when $g \in \Lambda$. Therefore, for the given $\mathbf{f} \in L^2(\Omega)$, there exist some positive constants $L_1 = L_1(m, M, \mathbf{f})$ such that

$$\| A_1^{\frac{1}{2}} P_1 \mathbf{u}_g \|^2 \leqslant \frac{2L_1}{m} \quad \text{for all } t \geqslant t_g, \quad g \in \Lambda.$$

Refer to Roh [7] for the details of L_1.

Definition 6. We define the set $U_s \subset V_1$ as

$$U_s = \left\{ \mathbf{v} \in V_1 \ : \ \| A_1^{\frac{1}{2}} \mathbf{v} \|^2 \leqslant \frac{2L_1}{m} \right\}.$$

Proof of Theorem II. First, for $g \in \Lambda$, one can easily see from the definition of $\tilde{\sigma}_s(g, \mathbf{v}, t)$ that the semiflow $\tilde{\sigma}_s(g, \mathbf{v}, t)$ on the space V_1 has a global attractor as we mentioned for the semiflow $\tilde{\sigma}_w(g, \mathbf{v}, t)$.

Also, by Lemmas 9 and 10, one note that $\tilde{\sigma}_s(1, \mathbf{v}, t)$ be imbedded into continuous family $\tilde{\sigma}_s(g, \mathbf{v}, t)$.

Moreover, by the definition of the set U_s, for $g \in \Lambda$ $\tilde{\sigma}_s(g, \mathbf{v}, t)$ is asymptotically compact on U_s because the global attractor of the semiflow $\tilde{\sigma}_s(g, \mathbf{v}, t)$ is contained in the set U_s.

Therefore, by the Robustness theorem, Proposition 9 that the semiflow $\tilde{\sigma}_s(g, \mathbf{v}, t)$ is robust at $\mathcal{A}_1$, the global attractor of $\tilde{\sigma}_s(1, \mathbf{v}, t) = \sigma_s(1, \mathbf{v}, t)$. Note that $\mathcal{A}_1$ is the global attractor of the 2D Navier–Stokes equations.

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