A Finite Element Gauge-Uzawa Method for the Evolution Navier-Stokes Equations

Jae-Hong Pyo [1]

1) *Department of Mathematics, Kangwon National University, Korea.*

ABSTRACT

The Navier-Stokes of incompressible fluids are still a computational challenge. The numerical difficulty arises from coupling variables for velocity and pressure, which requires a compatibility condition (discrete inf-sup) between the finite element spaces. Several projection methods [2,5,7]have been introduced for time discretization to circumvent the incompressibility constraint, but suffer from boundary layers. They are either numerical or due to non-physical boundary conditions on pressure.

We introduce a first order gauge-Uzawa method in [1,3,5] for time discretization coupled with a stable finite elements method for space discretization. The method is unconditionally stable, consists of $d+1$ Poisson solvers per time steps, and does not exhibit pronounced boundary layer effects. We prove error estimates for both velocity and pressure under realistic regularity conditions via a variational approach, and illustrate the performance with several numerical experiments. Additionally, we apply the gauge-Uzawa into more complicate problems which are Boussineq equations [4] and the non-steady density Navier-Stokes equations[6].

GAUGE-UZAWA FINITE ELEMENT METHOD(GU-FEM)

GU-FEM [3,5] for the Navier-Stokes equations can be written as following:

Start with $s_h^0 = 0$ and $\mathbf{u}_h^0$ as a solution of $\langle \mathbf{u}_h^0 \,,\, \mathbf{w}_h \rangle = \langle \mathbf{u}^0 \,,\, \mathbf{w}_h \rangle$ for all $\mathbf{w}_h \in \mathbb{V}_h$.

Step 1: Find $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$ as the solution of

$$\begin{aligned}\tau^{-1}\left\langle \widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n \,,\, \mathbf{w}_h \right\rangle + \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \mu \left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle \\ - \mu \left\langle s_h^n \,,\, \operatorname{div} \mathbf{w}_h \right\rangle = \left\langle \mathbf{f}(t^{n+1}) \,,\, \mathbf{w}_h \right\rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h.\end{aligned} \tag{1}$$

Step 2: Find $\rho_h^{n+1} \in \mathbb{P}_h$ as the solution of

$$\left\langle \nabla \rho_h^{n+1} \,,\, \nabla \psi_h \right\rangle = \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \,,\, \psi_h \right\rangle, \qquad \forall \psi_h \in \mathbb{P}_h. \tag{2}$$

Step 3: Update $s_h^{n+1} \in \mathbb{P}_h$ according to

$$\left\langle s_h^{n+1} \,,\, q_h \right\rangle = \left\langle s_h^n \,,\, q_h \right\rangle - \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \,,\, q_h \right\rangle, \qquad \forall q_h \in \mathbb{P}_h. \tag{3}$$

Step 4: Update $\mathbf{u}_h^{n+1} \in \mathbb{V}_h + \nabla \mathbb{P}_h$ according to

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla \rho_h^{n+1}. \tag{4}$$

Whenever necessary, one may seek pressure as compute

$$p_h^{n+1} = \mu s_h^{n+1} - \tau^{-1} \rho_h^{n+1}. \tag{5}$$

THE MAIN RESULTS

We now summarize theoretical results of GU-FEM [3]. The GU-FEM is **unconditionally stable** in the sense that, for all $\tau > 0$, the following a priori bound holds:

$$\begin{aligned} &\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N} \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n}\right\|_0^2 + \frac{\mu\tau}{2} \sum_{n=0}^{N} \left\|\nabla \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ &\quad + 2 \sum_{n=0}^{N} \left\|\nabla \rho_h^{n+1}\right\|_0^2 + \mu\tau \left\|s_h^{N+1}\right\|_0^2 \le \left\|\mathbf{u}_h^0\right\|_0^2 + C\tau \sum_{n=0}^{N} \left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2, \end{aligned} \tag{6}$$

and the GU-FEM hold following error bound with realistic regularity assumptions:

$$\begin{aligned} \tau \sum_{n=0}^{N} \left(\left\| \nabla \left(\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1} \right) \right\|_0^2 + \left\| p(t^{n+1}) - p_h^{n+1} \right\|_0^2 \right) &\le C(\tau + h^2), \\ \tau \sum_{n=0}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1} \right\|_0^2 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1} \right\|_0^2 \right) &\le C(\tau + h^2)^2. \end{aligned}$$

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